$$= \int_0^1 12 y_1^2 (1 - 2y_1 + y_1^2)\, dy_1$$

$$= \int_0^1 12 (y_1^2 - 2y_1^3 + y_1^4)\, dy_1 = 12\left(\frac{y_1^3}{3} - \frac{2y_1^4}{4} + \frac{y_1^5}{5}\right)\Big|_0^1$$

hard way

$$= 12\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = 12\ \frac{10-15+6}{30} = \frac{12}{30} = \frac{2}{5}$$

It is usually easier to find the marginal:

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^{1-y_1} 24\, y_1 y_2\, dy_2$$

$$= 24 y_1 \frac{y_2^2}{2}\Big|_{y_2=0}^{y_2=1-y_1} = 12\, y_1 (1-y_1)^2,\ 0<y_1<1$$

easy way

$$\text{so } E(Y_1) = \int_0^1 y_1\, 12\, y_1 (1-y_1)^2\, dy_1$$

$$= \int_0^1 12\, y_1^2 (1-y_1)^2\, dy_1 = \frac{2}{5}$$ by the above calculation.

OR $E[Y_1] =$

$$\int_0^1 \int_0^{1-y_2} y_1\, 24\, y_1 y_2\, dy_1\, dy_2$$

$$= \int_0^1 \int_0^{1-y_2} 24 y_1^2 y_2\, dy_1\, dy_2 = \int_0^1 24 \frac{y_1^3}{3} y_2 \Big|_{y_1=0}^{y_1=1-y_2} dy_2$$

$$= \int_0^1 8\, (1-y_2)^3 y_2\, dy_2$$

and the integral will take a longtime.

35.5

NOW $E(Y_1 Y_2) =$

$$\int_0^1 \int_0^{1-y_1} y_1 y_2 \, 24 \, y_1 y_2 \, dy_2 \, dy_1$$

$$= \int_0^1 \int_0^{1-y_1} 24 \, y_1^2 y_2^2 \, dy_2 \, dy_1$$

$$= \int_0^1 \left[24 \, y_1^2 \, \frac{y_2^3}{3} \Big|_0^{1-y_1} \right] dy_1$$

$$= \int_0^1 8 \, y_1^2 \, (1-y_1)^3 \, dy_1$$

$$= \int_0^1 8 y_1^2 \, (1-2y_1+y_1^2)(1-y_1) \, dy_1$$

$$= \int_0^1 8 y_1^2 \left[1 - 2y_1 + y_1^2 - y_1 + 2y_1^2 - y_1^3 \right] dy_1$$

$$= \int_0^1 8 y_1^2 \left[1 - 3y_1 + 3y_1^2 - y_1^3 \right] dy_1$$

$$= \int_0^1 8 \left[y_1^2 - 3y_1^3 + 3y_1^4 - y_1^5 \right] dy_1$$

$$= 8 \left(\frac{y_1^3}{3} - 3\frac{y_1^4}{4} + 3\frac{y_1^5}{5} - \frac{y_1^6}{6} \right) \Big|_0^1$$

$$= 8 \left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6} \right) = 8 \, \frac{20-45+36-10}{60}$$

$$= \frac{8}{60} = \frac{2}{15} = E \, Y_1 Y_2 .$$

SO $\mathrm{COV}(Y_1, Y_2) = E(Y_1 Y_2) - EY_1 \, EY_2$

$$= \frac{2}{15} - \frac{2}{5}\frac{2}{5} = \frac{10-12}{75} = \frac{-2}{75}$$

ex) Joint distribution of Y_1 and Y_2:

Y_1 \ Y_2	0	100	200	$p(Y_1=y_1)$
100	.2	.1	.2	.5
250	.05	.15	.3	.5
$p(Y_2=y_2)$	.25	.25	.5	1.0

So $E(Y_1) = 100(.5) + 250(.5) = \frac{350}{2} = 175$

$E(Y_2) = 0(.25) + 100(.25) + 200(.5) = 125$

$$E(Y_1 Y_2) = \sum_{(y_1,y_2):\, p(y_1,y_2)>0} \sum y_1 y_2 \, p(y_1,y_2)$$

$= 100(0)(.2) + (100)(100)(.1) + 100(200)(.2)$

$+ 250(0)(.05) + (250)(100)(.15) + (250)(200)(.3)$

$= 0 + 1000 + 4000 +$
$0 + 3750 + 15000$
$= 23750.$

So $Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$

$= 23750 - (175)(125) = 23750 - 21875$

$= 1875.$

Note that $E(Y_1 - Y_2) = EY_1 - EY_2$

$= 175 - 125 = 50.$

§5.8 42] Know p 271 Let $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_m$ be RV's.

Let $U_1 = \sum_{i=1}^{n} a_i Y_i$ and $U_2 = \sum_{i=1}^{m} b_i X_i$ for constants $a_1, \ldots, a_n$, $b_1, \ldots, b_m$.

a) $E(U_1) = E\left[\sum_{i=1}^{n} a_i Y_i\right] = \sum_{i=1}^{n} a_i E(Y_i)$

b) $V(U_1) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum\sum_{i<j} a_i a_j \operatorname{Cov}(Y_i, Y_j)$

where the sum is over all pairs (i,j) with $i<j$. $V(U_1) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j \operatorname{cov}(Y_i, Y_j)$

c) $\operatorname{Cov}(U_1, U_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \operatorname{Cov}(Y_i, X_j)$

proof of b) $V(U_1) = E(U_1 - E(U_1))^2 =$

$$E\left(\sum_{i=1}^{n} a_i Y_i - \sum_{i=1}^{n} a_i E Y_i\right)^2 =$$

$$E\left[\sum_{i=1}^{n} a_i (Y_i - EY_i)\right]^2 = E\left[\sum_{i=1}^{n} a_i (Y_i - EY_i) \sum_{j=1}^{n} a_j (Y_j - EY_j)\right]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j (Y_i - EY_i)(Y_j - EY_j)\right]$$

$$= E\left[\sum_{i=1}^{n} a_i^2 (Y_i - EY_i)^2 + \sum\sum_{i \neq j} a_i a_j (Y_i - EY_i)(Y_j - EY_j)\right]$$

$$= \sum_{i=1}^{n} a_i^2 V(Y_i) + \sum\sum_{i \neq j} a_i a_j \operatorname{cov}(Y_i, Y_j)$$

$$= \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum\sum_{i<j} a_i a_j \operatorname{cov}(Y_i, Y_j)$$

The last inequality follows since $cov(Y_i, Y_j) = cov(Y_j, Y_i)$.

and

	Y_1	Y_2	Y_3	$\cdots$	Y_{n-1}	Y_n
Y_1		$cov(Y_1,Y_2)$	$cov(Y_1,Y_3)$	$\cdots$	$cov(Y_1,Y_{n-1})$	$cov(Y_1,Y_n)$
Y_2	$cov(Y_2,Y_1)$		$cov(Y_2,Y_3)$		$cov(Y_2,Y_{n-1})$	$cov(Y_2,Y_n)$
Y_3	$cov(Y_3,Y_1)$	$cov(Y_3,Y_2)$		$\cdots$	$cov(Y_3,Y_{n-1})$	$cov(Y_3,Y_n)$
$\vdots$						$\vdots$
Y_{n-1}	$cov(Y_{n-1},Y_1)$	$cov(Y_{n-1},Y_2)$	$cov(Y_{n-1},Y_3)$	$\cdots$		$cov(Y_{n-1},Y_n)$
Y_n	$cov(Y_n,Y_1)$	$cov(Y_n,Y_2)$	$cov(Y_n,Y_3)$	$\cdots$	$cov(Y_n,Y_{n-1})$	

these sum to $\sum\sum_{i>j} cov(Y_i, Y_j)$

these sum to $\sum\sum_{i<j} cov(Y_i, Y_j)$

So $$\sum\sum_{i \neq j} a_i a_j cov(Y_i, Y_j) = 2 \sum\sum_{i>j} a_i a_j cov(Y_i, Y_j)$$

$$= 2 \sum\sum_{i<j} a_i a_j cov(Y_i, Y_j)$$

proof of c) $cov(U_1, U_2) = E(U_1 - EU_1)(U_2 - EU_2)$

$$= E\left[\left(\sum_{i=1}^{n} a_i Y_i - \sum_{i=1}^{n} a_i EY_i\right)\left(\sum_{j=1}^{m} b_j X_j - \sum_{j=1}^{m} b_j EX_j\right)\right]$$

$$= E\left[\sum_{i=1}^{n} a_i (Y_i - EY_i) \sum_{j=1}^{m} b_j (X_j - EX_j)\right]$$

$$= E \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j (Y_i - EY_i)(X_j - EX_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j E\left[(Y_i - EY_i)(X_j - EX_j)\right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j cov(Y_i, X_j).$$

43] Know p9 and p274 Let $Y_1, \ldots, Y_n$ be RVs. 37.5

The sample mean $\underbrace{\overline{Y}}_{\text{Ybar}} = \frac{1}{n}\sum_{i=1}^{n} Y_i$.

44] Know If $Y_1, \ldots, Y_n$ are independent, $EY_i = \mu$ and $V(Y_i) = \sigma^2$, then $E[\overline{Y}] = \mu$ and $V(\overline{Y}) = \frac{\sigma^2}{n}$.

proof] $E\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} EY_i = \frac{1}{n} n\mu = \mu$

$V(\overline{Y}) = V\left(\sum_{i=1}^{n} \frac{Y_i}{n}\right) \underset{\text{ind}}{=} \sum V\left(\frac{Y_i}{n}\right) + 0$

$= \frac{1}{n^2}\sum V Y_i = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

ch6] 1] Know $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed (iid) if the RVs are independent from the same distribution ie $f_{Y_i}(y) \equiv f_Y(y)$ or $p_{Y_i}(y) \equiv p(y)$.

ex] $Y_1, \ldots, Y_n$ are iid $N(0,1)$ means $Y_1, \ldots, Y_n$ are independent and Y_i is normal with mean 0 and variance 1 for $i = 1, \ldots, n$.

2] know p297 If $Y_1, \ldots, Y_n$ are iid, then $Y_1, \ldots, Y_n$ are a random sample.

A random sample from a pop with pdf $f(y)$ or prob function $p(y)$ means $f_{Y_i}(y_i) = f(y_i)$ or $p_{Y_i}(y_i) = p(y_i)$ $i=1,\ldots,n$.

3] If $Y_1, \ldots, Y_n$ are a random sample, then $f(y_1, \ldots, y_n) = f(y_1)\, f(y_2) \cdots f(y_n)$ or $p(y_1, \ldots, y_n) = p(y_1)\, p(y_2) \cdots p(y_n)$.

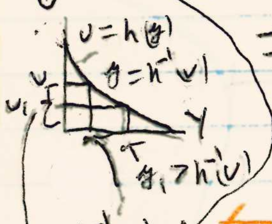
§6.2

4] Let $U = h(Y)$ be a function of Y where the distribution of Y is known. Goal: Find the distribution of U.

§6.3 Method of Distribution Functions

5] We are going to use the method of distribution functions if $U = h(Y)$ where h is a simple function eg decreasing or increasing function.

Idea: $F_U(u) = P(U \le u) = P(h(Y) \le u)$



$$= \begin{cases} P(Y \le h^{-1}(u)) = F_Y(h^{-1}(u)), & h \text{ increasing} \\ P(Y \ge h^{-1}(u)) = 1 - P(Y < h^{-1}(u)), & h \text{ decreasing} \end{cases}$$

← see notes 38½

$\{U \le u\} = \{Y \ge h^{-1}(u)\}$

Read ex 6.1 skip ex 6.2, 6.3, read p304 – 307.

38.5

ex) $U = Y^2$ $\underbrace{f(y) = \frac{1}{2}, \quad -1 \le y \le 1}^{\text{uniform } (-1,1)}$

since $-1 \le Y \le 1$, $0 \le U \le 1$

$F_U(u) = P(U \le u) = P(Y^2 \le u)$

$= P(-\sqrt{u} \le Y \le \sqrt{u})$

$= \int_{-u^{1/2}}^{u^{1/2}} \frac{1}{2}\, dy = \frac{y}{2}\Big|_{-u^{1/2}}^{u^{1/2}}$

$= \frac{u^{1/2}}{2} - -\frac{u^{1/2}}{2} = u^{1/2}$ for $0 \le u \le 1$

$F_U(u) = u^{1/2}$, $0 \le u \le 1$. So $f_U(u) =$

$\frac{d}{du} F_U(u) = \frac{1}{2} u^{1/2 - 1} = \frac{1}{2u^{1/2}}$ for $0 < u \le 1$.

ex) $U = 3Y$, $f(y) = \frac{1}{2}$, $-1 \le y \le 1$

since $-1 \le y \le 1$, $-3 \le u \le 3$.

$F_U(u) = P(U \le u) = P(3Y \le u) =$

$P(Y \le \frac{u}{3}) = F_Y(\frac{u}{3}) = \int_{-1}^{u/3} \frac{1}{2}\, dy$

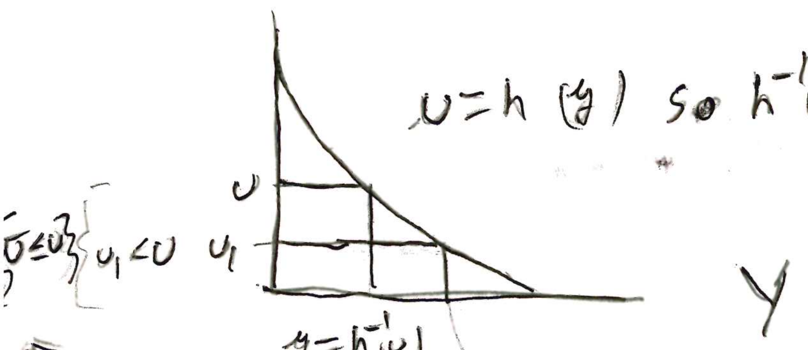
$= \frac{1}{2} y\Big|_{-1}^{u/3} = \frac{1}{2}\left(\frac{u}{3} + \underbrace{\frac{3}{3}}_{1}\right) = \frac{u+3}{6}$.

So $F_U(u) = \frac{u+3}{6}$, $-3 \le u \le 3$

$f_U(u) = \frac{d}{du} F_U(u) = \underbrace{\frac{1}{6} \quad -3 \le u \le 3}_{\text{Uniform } (-3,3)}$

h decreasing

$u = h(y)$ so $h^{-1}(u) = y$



$y_1 = h^{-1}(u_1) > h^{-1}(u)$ if $u_1 < u$

inequality switches $\rightarrow$ $\{Y \ge h^{-1}(u)\}$

$$\{U \le u\} = \{Y \ge h^{-1}(u)\}$$

$$= \{\omega \in \Omega : U(\omega) \le u\}$$

mnemonic

$U = -Y$

so $P(U \le u)$

$= P(-Y \le u)$

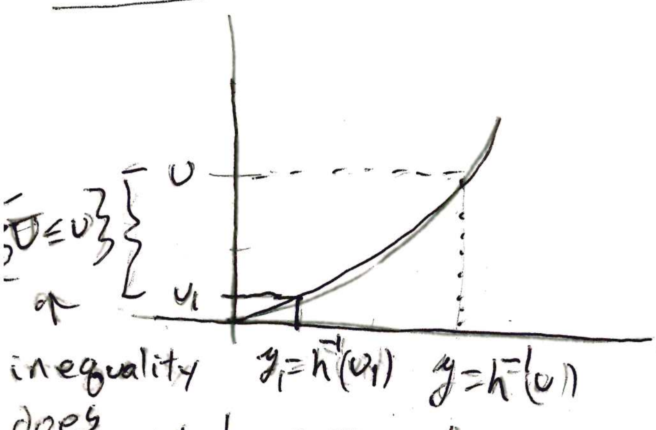
$= P(Y \ge u)$

For decreasing function, when taking the inverse function of both sides, switch the inequality

↑ do

↓ omit

h increasing



inequality does not switch $\rightarrow$ $\{U \le u\} = \{Y \le h^{-1}(u)\}$

ex] $U = -Y - 4$ $\quad f(y) = \frac{1}{2}, \ -1 \le y \le 1$

$U = -Y - 4$ so $-5 \le u \le -3$.

$F_U(u) = P(U \le u) = P(-Y - 4 \le u) = P(-Y \le u + 4)$

$= P(Y \ge -u - 4) = \int_{-u-4}^{1} \frac{1}{2}\, dy$

$= \frac{1}{2} y \Big|_{-u-4}^{1} = \frac{1}{2}(1 - (-u-4)) = \frac{1}{2}(5+u), \ -5 \le u \le -3$

so $f_U(u) = \frac{d}{du} F_U(u) = \underbrace{\frac{1}{2}, \quad -5 \le u \le -3}_{}$

U is uniform$(-5, -3)$.

6] Often if the support of Y is $a \le Y \le b$, then the support of $U = h(Y)$ can be found by solving

$a \le y \le b \iff a \le h^{-1}(u) \le b$

for U, h^{-1} must exist

ex] In the last ex $U = h(Y) = -Y - 4, \ -1 \le y \le 1$

$h(-1) = -(-1) - 4 = -3, \quad h(1) = -1 - 4 = -5$

so $-5 \le u \le -3$

If $a = -\infty$ or $b = \infty$, $h(a) = \lim_{y \downarrow -\infty} h(y)$

$h(b) = \lim_{y \uparrow \infty} h(y)$.

easier{ $h(a) \le u \le h(b)$, $h \uparrow$ $\qquad h(b) \le u \le h(a)$ $h \downarrow$

§6.4?} p313 Know Method of Transformations

If Y has pdf $f_Y(y)$ and $h(y)$ is either increasing or decreasing for all $\underbrace{y \text{ such that } f_Y(y) > 0}_{\text{support of } Y}$, then

$U = h(Y)$ has pdf $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{d h^{-1}(u)}{du} \right|$

on the support of U.

Skip ex 6.8 and 6.9 p 314 - 315 39.5

ex] Let $f_Y(y) = \dfrac{1}{y\sqrt{2\pi\sigma^2}} \exp\left[-\dfrac{(\log(y)-\mu)^2}{2\sigma^2}\right], \; y>0$

$\sigma > 0, \; \mu \in \mathbb{R}$.

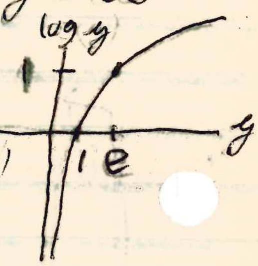
Unless otherwise stated $\log(y)$ is the natural logarithm.

Let $U = \log(Y)$ and find the distribution of U

step i) Find the support of U. $0 < y < \infty$

$u = \log(y) = h(y)$ so $-\infty < U < \infty$

"$\log(0)$" $= \lim_{y\to 0} \log(y) = -\infty$; "$\log(\infty)$" $= \lim_{y\to\infty} \log(y) = \infty$



Solve $U = h(y)$ for $y = h^{-1}(u)$:

step ii) Find $y = h^{-1}(u)$

$e^u = e^{\log(y)} = y$ so $y = h^{-1}(u) = e^u$

step iii) $\frac{d}{du} h^{-1}(u) = \frac{d}{du} e^u = e^u$, $|e^u| = e^u$

step iv) Use the formula $f_U(u) = f_Y(h^{-1}(u)) \left|\dfrac{d(h^{-1}(u))}{du}\right|$

$$= \frac{1}{e^u\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(\log(e^u)-\mu)^2}{2\sigma^2}\right) e^u$$

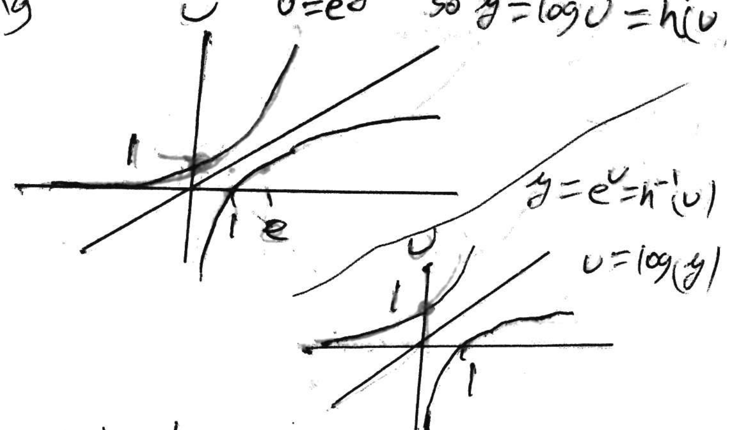
$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(u-\mu)^2}{2\sigma^2}\right], \; -\infty < u < \infty.$$

So U is normal with mean μ and variance σ^2.

Idea of method of transformations

i) Graphically obtain $y = h^{-1}(u)$ by reflecting $u = h(y)$ about the $u = y$ line.

Hence if h is increasing (decreasing), so is $h^{-1}(u)$.



ii) If $h^{-1}\uparrow$, then $\frac{dh^{-1}}{du} \geq 0$

so $\left|\frac{dh^{-1}(u)}{du}\right| = \frac{dh^{-1}(u)}{du}$.

If $h^{-1}\downarrow$, then $\frac{dh^{-1}}{du} \leq 0$ so $-\frac{dh^{-1}(u)}{du} = \left|\frac{dh^{-1}(u)}{du}\right|$

iii) Chain rule: if $h\uparrow$, then $h^{-1}\uparrow$ and

$F_U(u) = P(h(Y) \leq u) = P[Y \leq h^{-1}(u)] = F_Y(h^{-1}(u))$.

so $f_U(u) = \frac{d}{du} F_Y(h^{-1}(u)) = f_Y(h^{-1}(u)) \frac{dh^{-1}(u)}{du}$

$= f_Y(h^{-1}(u)) \left|\frac{dh^{-1}(u)}{du}\right|$.

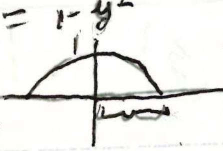
If $h\downarrow$, then $h^{-1}\downarrow$ and $F_U(u) = P(h(Y) \leq u)$

$= P(Y \geq h^{-1}(u)) \underset{\text{contin RV}}{\overset{\text{as}}{=}} 1 - P[Y \leq h^{-1}(u)]$

$= 1 - F_Y(h^{-1}(u))$. So $f_U(u) = \frac{d}{du} F_U(u) = -f_Y(h^{-1}(u)) \frac{d(h^{-1}(u))}{du}$

$= f_Y(h^{-1}(u)) \left|\frac{dh^{-1}(u)}{du}\right|$.

ex] $f_Y(y) = 3y^2, \; 0<y<1$. Find the pdf of $U = 1-Y^2$.



h is decreasing on $(0,1)$

$u = 1-y^2 = h(y), \quad h(0)=1, \quad h(1)=0$

i) $0 \le y \le 1$ so $0 \le u \le 1$ is the support of U.

solve $u = h(y)$ for $y = h^{-1}(u)$

ii) $u = h(y) = 1-y^2, \quad y^2 = 1-u, \quad y = \sqrt{1-u} = h^{-1}(u)$.

iii) $\frac{d}{du}(1-u)^{\frac{1}{2}} = -\frac{1}{2}(1-u)^{-1/2} = \frac{d}{du}h^{-1}(u)$

$$\left|\frac{d}{du}h^{-1}(u)\right| = \frac{1}{2\sqrt{1-u}}$$

iv) $f_U(u) = f_Y(h^{-1}(u))\left|\frac{d}{du}h^{-1}(u)\right|$

$$= 3\left(\sqrt{1-u}\right)^2 \frac{1}{2\sqrt{1-u}} = \frac{3}{2}\sqrt{1-u}, \quad 0<u<1.$$

8] common final problem: Find the pdf of $U = h(Y)$ where the pdf of Y is known. Usually $h(y) = y^2$, h is increasing, or h is decreasing. Use method of transformations (or method of distribution functions).

not in book

9] common problem: Find the probability function of $U = h(Y)$ given the probability function $p_Y(y)$ in a table.

step i) compute $h(y)$ for each y.

step ii) Collect $y: h(y) = u$ and sum the corresponding probabilities.

40.5

ex)

y	-2	-1	0	1	2
$p_Y(y)$	.1	.2	.3	.12	.28

Find $p_U(u)$ if $U = Y^2$.

step 1)

y^2	4	1	0	1	4
$P(Y=y)$	.1	.2	.3	.12	.28

step 2)

U	0	1	4
if	$y=0$	$y=-1,1$	$y=-2,2$
with prob	.3	$.2+.12$	$.1+.28$

U	0	1	4
$p_U(u)$	.3	.32	.38

§6.5

10) p318 If X and Y have the same mgf's: $m_X(t) = m_Y(t)$ for all t, then X and Y have the same probability distribution.

11) Let Y_i have mgf $m_{Y_i}(t)$, let $Y_1, \ldots, Y_n$ be independent, and let $U = \underbrace{\sum_{i=1}^{n} Y_i}_{\text{sum}} = Y_1 + \cdots + Y_n$.

Then the mgf of U is

$$m_U(t) = \underbrace{\prod_{i=1}^{n} m_{Y_i}(t)}_{\text{Product}} = m_{Y_1}(t)\, m_{Y_2}(t) \cdots m_{Y_n}(t).$$

4834/

proof] Know $m_U(t) = E[e^{tU}]$

$= E[e^{t(Y_1+\dots+Y_n)}] = E[e^{tY_1} e^{tY_2} \dots e^{tY_n}]$

$\overset{ind}{=} E(e^{tY_1}) E(e^{tY_2}) \dots E(e^{tY_n})$

$= m_{Y_1}(t) \dots m_{Y_n}(t)$.

12] Know p 320 Let $Y_1, Y_2, \dots, Y_n$ be independent normal RV's with $EY_i = \mu_i$ and $V(Y_i) = \sigma_i^2$, $i=1,\dots,n$.

Let $a_1, a_2, \dots, a_n$ be constants.

If $U = \sum_{i=1}^n a_i Y_i = a_1 Y_1 + \dots + a_n Y_n$, then U is normal with

$EU = \sum_{i=1}^n a_i \mu_i = a_1\mu_1 + \dots + a_n \mu_n$

and $V(U) = \sum_{i=1}^n a_i^2 \sigma_i^2 = a_1^2\sigma_1^2 + \dots + a_n^2 \sigma_n^2$

proof] If W is normal with mean μ and variance σ^2, then the mgf of W is $m_W(t) = \exp\left(t\mu + \frac{t^2}{2}\sigma^2\right)$, $t \in \mathbb{R}$. *

so $m_{a_iY_i}(t) = E(e^{ta_iY_i}) = m_{Y_i}(ta_i)$

$= \exp\left(ta_i\mu_i + \frac{t^2 a_i^2}{2} \sigma_i^2 \right)$.

$m_U(t) = m_{\Sigma a_i Y_i}(t) = \prod_{i=1}^n m_{a_iY_i}(t) =$

$\exp\left(t\mu_1 a_1 + \frac{t^2}{2}\sigma_1^2 a_1^2\right) \dots \exp\left(t\mu_n a_n + \frac{t^2}{2}\sigma_n^2 a_n^2\right)$

$= \exp\left[\sum t a_i \mu_i + \frac{t^2}{2} a_i^2 \sigma_i^2\right]$

$= \exp\left[t \underbrace{\sum_{i=1}^{n} a_i \mu_i}_{\mu_U} + \frac{t^2}{2} \underbrace{\sum a_i^2 \sigma_i^2}_{\sigma_U^2}\right]$

which is the mgf of a normal RV

with mean $\mu_U = \sum_{i=1}^{n} a_i \mu_i$ and

variance $\sigma_U^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$ . . (See $\underline{*}$.)

13] $\boxed{\text{Know}}$ If $Y_1, \ldots, Y_n$ are a random sample (iid) from a normal distribution with mean μ and variance σ^2, then

$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is normal with

$E(\bar{Y}) = \mu$ and $V(\bar{Y}) = \frac{\sigma^2}{n}$.

proof] Take $\sigma_i^2 = \sigma^2$, $\mu_i = \mu$, and $a_i = \frac{1}{n}$, $i = 1, \ldots, n$

in the last result.

14] * p 322 If $Y_1, \ldots, Y_n$ are independent normal RV's with $E Y_i = \mu_i$ and $V Y_i = \sigma_i^2$, then

$Z_i = \frac{Y_i - \mu_i}{\sigma_i}$ is normal 0, 1

and $W = \sum_{i=1}^{n} Z_i^2$ is χ_n^2.

15] Often the transformation $U = h(Y)$ is a gamma(α, β) distribution