Math 501 HW 10 Spring 2025. Due Friday, April 18. Exam 3 Friday, April 25: a change from the original syllabus.
Final Tuesday, May 6, 8-10AM.

Exam 2 and 3 reviews may be useful.

1) Suppose $f \geq 0$ is nonnegative. Find f^+ and f^-.
(Your result can be used to show that the definitions for integrability and for the integral for general functions agrees with the definitions for nonnegative functions when f is nonnegative.)

2) Suppose the function f satisfies $f(x) = 4$ for $x < 0$ and $f(x) = x^2$ for $x \geq 0$. Let $a = -4 = x_0 < x_1 < \cdots < x_k = 4 = b$ be a partition π of $[a, b] = [-4, 4]$. Let x_j be the smallest $x_i \geq 0$. Then $t = V_a^b(f, \pi) = \sum_{i=1}^k |f(x_i) - f(x_{i-1})| =$

$$\sum_{i=1}^{j-1} |f(x_i) - f(x_{i-1})| + |f(x_j) - f(x_{j-1})| + \sum_{i=j+1}^{k} |f(x_i) - f(x_{i-1})| = A + B + C.$$

Then $C = f(4) - f(x_j) = 16 - x_j^2$ since C is equal to the sum of the heights of an increasing step function from $x = x_j$ to $x = 4$. (You may want to sketch f.)
a) If $2 \leq x_j \leq 4$, find B.
b) If $0 \leq x_j < 2$, find B.
c) If $2 \leq x_j \leq 4$, find $A + B + C$.
d) If $0 \leq x_j < 2$, find $A + B + C$.
e) Which value of x_j maximizes $A + B + C$?
f) The $t = A + B + C$ corresponding to that x_j is equal to $T = \sup_\pi t$. What is T?
g) Is f of bounded variation on $[-4, 4]$?

3) Prove the Cauchy Schwarz Inequality using Holder's Inequality.

Hint the proof should be 1 sentence: The Cauchy Schwarz Inequality follows from Holder's Inequality with $p = q =$.

4) Prove that measure μ is finitely subadditive using the fact that μ is countably subadditive. So let $E_1, ..., E_N$ be measurable sets, and prove $\mu(\cup_{i=1}^N E_i) \leq \sum_{i=1}^N \mu(E_i)$.

5) Let A and B be measurable sets.
a) Prove $\mu(A) = \mu(A - B) + \mu(A \cap B)$.
b) Prove $\mu(A \cup B) = \mu(A - B) + \mu(A \cap B) + \mu(B - A)$.

Hint: Use the proof from HW 5 with m replaced by μ. A Venn diagram with disjoint sets may be useful. Avoid $\infty - \infty$.

6) Let $\mu(X) = 10$. Let $A, B \in \mathcal{F}$, $\mu(A) = 3$, and $\mu(A \cap B) = 1$. Let $\phi = 2\chi_A$. Let $\int f \, d\mu = 8$. For a) $-$ d) calculate

a) $\int \chi_A \, d\mu$

b) $\int \phi \, d\mu$

c) $\int (2f + 4\phi) \, d\mu$

d) $\mu(A - B)$.

e) Show $\mu(B) \geq 1$.

Hint: the answers are almost the same as for HW7 with $\int f = \int f \, d\mu$, so replace m by μ.