Math 501 HW 11 Spring 2025. Due Monday, April 28. Exam 3 Friday, April 25: a change from the original syllabus.
Final Tuesday, May 6, 8-10AM.

Exam 2 and 3 reviews may be useful.

1) Consider the Cartesian product of $[a,b]\times[c,d]$. Let the product measure $\pi = m\times m$ so that both $\mu$ and $\nu$ are L. measure. Find $\pi([a,b]\times[c,d])$.

2) Let $(X,\mathcal{F},m)$ be the measure space, and let $f \geq 0$ be a nonnegative measurable function. Let $E \in \mathcal{F}$ be such that $m(E) = 0$. Let $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$, otherwise. Assume that the $f_n$ and $f$ are defined on $E$. Then each $f_n$ is bounded, $f_n \leq f$, and $f_n(x) \uparrow f(x)$ for $x \in E$. Let $\epsilon > 0$. By the Monotone Convergence Theorem, there is an $N$ such that $\int_E f_N > \int_E f - \epsilon$, and $\int_E (f - f_N) < \epsilon$. Then $0 \leq \int_E f = \int_E (f - f_N) + \int_E f_N < \epsilon + Nm(E) = \epsilon$. Since $\epsilon > 0$ was arbitrary, $\int_E f = 0$. This result also holds if $m$ is replaced by $\mu$.

Define the measure $\nu(E) = \int_E f \; d\mu$ for $E \in \mathcal{F}$. Prove that if $\mu(E) = 0$, then $\nu(E) = 0$.

(Hence $\nu$ is absolutely continuous wrt $\mu$.)

3) What theorem is used to prove that double integrals can be computed with iterated integrals?

4) Let $(X,\mathcal{F},\mu)$ be the measure space, and let $f$ be an integrable function (integrable $\mu$). Let $E \in \mathcal{F}$ be such that $\mu(E) = 0$. Let $f_n(x) = f(x)$ if $-n \leq f(x) \leq n$, $f_n(x) = -n$ if $f(x) < -n$, and $f_n(x) = n$, if $f(x) > n$. Assume that the $f_n$ and $f$ are defined on $E$. Then each $f_n$ is measurable, bounded, $|f_n| \leq n$, $|f_n| \leq |f|$, and $f_n(x) \to f(x)$ for $x \in E$.
a) What theorem shows $\int_E f_n \; d\mu \to \int_E f \; d\mu$?
b) Prove that for each $n$, $\int_E f_n \; d\mu = 0$.
c) So what is the value of $\int_E f \; d\mu$?

5) Bounded Convergence Theorem (BCT): Let $E \in \mathcal{F}_M$, $m(E) < \infty$, and $f_n \in \mathcal{L}(E)$. If $\exists M > 0$ such that $|f_n(x)| \leq M$ $\forall n \in \mathbb{N}$ and $\forall x \in E$, and if $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$, then $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$.

Prove the BCT using the LDCT. Make sure that you show that $g$ or the $f_n$ are integrable.

6) Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose $E \in \mathcal{F}$ and $f : X \to [-\infty,\infty]$ is a extended real valued function such that $\int_E f \; d\mu$ is defined. Prove that

$$\left| \int_E f \; d\mu \right| \leq \mu(E) \sup_E |f|.$$