

Math 501 HW 2 Spring 2025. Due Friday, Jan. 31.

Place your solutions on a separate sheet of paper. DO NOT place solutions side by side. You may use both the front and the back of each sheet.

YOU ARE BEING GRADED FOR WORK NOT JUST THE FINAL ANSWER. As a rule of thumb, you should have some idea of what you were doing, even without the book or notes. You are encouraged to form groups to discuss ideas and HW problems, but do not copy.

Exam 1 review may be useful. For the quiz, the exam 1 review and qual problems from the course website may be useful. 3 sheets of notes for the quiz.

To prove that \mathcal{F} is a σ -algebra, show

0) \mathcal{F} is nonempty. Often this is done by showing that $X \in \mathcal{F}$.

i) $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

1) Let Λ be an arbitrary nonempty index set, and for $\lambda \in \Lambda$, let \mathcal{F}_λ be a σ -algebra on X . Prove that $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a σ -algebra on X .

Hint: to prove that \mathcal{F} is nonempty, show that $X \in \mathcal{F}$.

2) Let \mathcal{A} be a class of subsets of X . The σ -algebra generated by \mathcal{A} , denoted by $\sigma(\mathcal{A})$, is the intersection of all σ -algebras containing \mathcal{A} . Thus $\sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ where Λ is the collection of σ -algebras \mathcal{F}_λ that contain \mathcal{A} . Prove that $\sigma(\mathcal{A})$ is a σ -algebra.

Hint: See problem 1 on the qual problems from the course webpage and notes 48). This problem is similar to (R. #19, p. 19).

3) Let $\mathcal{A} = \{\emptyset\}$. Prove that $\sigma(\mathcal{A}) = \{\emptyset, X\}$ (which is the smallest σ -algebra).

4) (R. #11, p. 16): Prove that $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

5) Simplify the following sets. Answers might be $(a, b), [a, b), (a, b], [a, b], [a, a] = \{a\}, (a, a) = \emptyset$.

i) $\bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n} \right] =$

ii) $\bigcup_{n=1}^{\infty} \left[a, b - \frac{1}{n} \right] =$