Math 501 HW 3 Spring 2025. Due Friday, Feb. 7. **Problem 5 is on two pages.**

Exam 1 review may be useful. For the quiz, the exam 1 review and qual problems from the course website may be useful. 3 sheets of notes for the quiz.

1) Consider proving that a countable union of countable sets is countable. Consider a countable number of sets A_1, A_2, ... where each set A_i is countable with $A_1 = \{a_{11}, a_{21}, a_{31}, ...\}$, $A_i = \{a_{1i}, a_{2i}, a_{3i}, ...\}$. Write the elements of the countable union of countable sets in the following form.

$$\begin{array}{ccccc} a_{11} & a_{21} & a_{31} & a_{41} & \cdots \\ a_{12} & a_{22} & a_{32} & a_{42} & \cdots \\ a_{13} & a_{23} & a_{33} & a_{43} & \cdots \\ a_{14} & a_{24} & a_{34} & a_{44} & \cdots \\ a_{15} & \cdots & & & \\ \vdots & & & & \end{array}$$

Copy these elements on a piece of paper and draw in the zigzag map that gives the set $\{a_{11}, a_{21}, a_{12}, a_{13}, a_{22}, a_{31}, a_{41}, a_{32}, ..., a_{42}...\}$. The first element has the sum of subscripts equal to 2, the next two elements have the sum of subscripts equal to 3, the next three elements have the sum of subscripts equal to 4, etc. (This map with domain $\mathbb{N}$ is onto the elements of the countable union of countable sets. Thus the required theorem is "proved," ignoring that some of the A_i may be finite or equal to the empty set).

2) Let $\mathcal{A}$ be the class of all open and closed sets contained in $\mathbb{R}$. Hence if $B \in \mathcal{A}$, then either B is an open set or B is a closed set with $B \subseteq \mathbb{R}$. Prove that $\mathcal{A}$ is not an algebra. (Hence $\mathcal{A}$ is not a σ-algebra.)

3) (R. #20, p. 40): Let $\{x_n\}$ be a sequence of real numbers. Show that $x = \lim_{n\to\infty} x_n$ iff

$$x = x_1 + \sum_{i=1}^{\infty}(x_{i+1} - x_i).$$

Hint: Show $x = x_1 + \sum_{i=1}^{\infty}(x_{i+1} - x_i) = x_1 + \lim_{n\to\infty}(\sum_{i=1}^{n}(x_{i+1} - x_i)) = \cdots = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$. Write the last sum as $(x_2 - x_1 + x_3 - x_2 + \cdots + x_n - x_{n-1} + x_{n+1} - x_n)$ and all but 1 of the terms cancel.

4) (R. #14, p. 39): Show that $\overline{lim}_n\, x_n = \infty$ iff given Δ and n, $\exists k \geq n$ with $x_k > \Delta$. Hint: Show LHS $\Rightarrow$ RHS and show RHS $\Rightarrow$ LHS. Use $\overline{lim}_n\, x_n = \inf_n\, \sup_{k\geq n} x_k$.

5) To prove that the set of all real numbers in $[0, 1]$ is not countable, do the following steps. First, every real number in $[0, 1]$ has a decimal expansion $0.a_1a_2a_3...$ where $a_1, a_2, ...$ are any of the digits 0,1,...,9.

We assume that numbers whose decimal expansions terminate, such as 0.7234, are written as 0.73240000... and that this is the same as 0.7239999....

If all real numbers in $[0, 1]$ are countable, then we can place them in 1–1 correspondence with the natural numbers as follows:

1: $0.a_{11}a_{12}a_{13}a_{14}...$

2: $0.a_{21}a_{22}a_{23}a_{24}...$

3: $0.a_{31}a_{32}a_{33}a_{34}...$

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Now form a number $0.b_1b_2b_3b_4...$ where $b_1 = 6$ if $a_{11} = 5$ and $b_1 = 5$ if $a_{11} \neq 5$, $b_2 = 6$ if $a_{22} = 5$ and $b_2 = 5$ if $a_{22} \neq 5$, etc. (The choice of 5 and 6 could be replaced with by two other numbers, and binary expansions are popular.)

Now explain why $[0, 1]$ is not countable.

(This technique is known at the diagonal technique.)