Math 501 HW 5 Spring 2025. Due Friday, Feb. 28.

Exam 2 review may be useful.

1) Let $A_1, A_2, ...$ be subsets of $\mathbb{R}$ such that the outer measure $m^*(A_i) = 0$ for $i = 1, 2, ...$ Prove that the measure $m(\cup_{i=1}^{\infty} A_i) = 0$.

Hint: $A_1, A_2, ... \in \mathcal{F}_M$. Thus $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_M$. (Write these facts on your homework solution.)

2) Let A and B be measurable sets (with $X = \mathbb{R}$).

a) Prove $m(A) = m(A - B) + m(A \cap B)$.

b) Prove $m(A \cup B) = m(A - B) + m(A \cap B) + m(B - A)$.

c) Under what conditions is it true that $m(A \cup B) = m(A) + m(B) - m(A \cap B)$?

Hint: A Venn diagram with disjoint sets may be useful. Avoid $\infty - \infty$.

3) (R. #8, p.58): Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Hint: Use monotonicity ($B \subseteq A \cup B$) and subadditivity.

4) Almost (R. #10, p.64): Let E_1 and E_2 be measurable sets. Then $E_1 \cup E_2 = E_1 \cup (E_2 \cap E_1^c)$ is a union of disjoint measurable sets. Use this fact to prove $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$. (So do not use problem 2 to prove this result.)

5) Almost (R. #11, p. 64): Prop. 14: Let $E_1, E_2...$ be an infinite decreasing sequence of measurable sets: $E_{n+1} \subseteq E_n \forall n$. Let $m(E_1)$ be finite. Then $m(\cap_{i=1} E_i) = \lim_{n\to\infty} m(E_n)$.

Let $E_n = (n, \infty)$ to show that $m(E_1) < \infty$ is necessary for Prop. 14 to hold.