Math 501 HW 6 Spring 2025. Due Friday, March 7.

Exam 2 review may be useful. **4 problems on two pages**

1) (R. #20, p. 70, modified): Show a) $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$.
b) Show $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_A(x)\chi_B(x)$.
c) Show $\chi_{A^c}(x) = 1 - \chi_A(x)$.

Hint: Figure out where the LHS = 0 or 1 and figure out where the RHS = 0 or 1. Recall that $\chi_A(x) = I_A(x)$.

2) (R. #20, p. 70, modified): Let $\phi(x) = \sum_{i=1}^n \alpha_i \chi_{A_i}(x)$ and $\psi(x) = \sum_{j=1}^m \beta_j \chi_{B_j}(x)$ where the α_i and β_j are real and the A_i and B_j are measurable. Let $A = \cup_{i=1}^n A_i$ and $B = \cup_{j=1}^m B_j$.

a) Show that the product $(\phi(x))(\psi(x)) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j \chi_{E_{ij}}(x)$ and find the measurable sets E_{ij}.

This result shows that if simple functions ϕ and ψ are measurable, then the product of the two functions is a simple function that is measurable. Problem 1 will be useful.

b) Simplify $\phi(x) + \psi(x)$ for i) $x \in A - B$, ii) $x \in B - A$, iii) $x \in A \cap B$, and iv) $x \in A^c \cap B^c$.

This result shows that if simple functions ϕ and ψ are measurable, then the sum of the two functions is a simple function that is measurable (since the sum assumes only a finite number of values). One of the terms can't be simplified much.

3) (R. #21 a) , p. 70): Let D and E be measurable, and let f be a function with domain $D \cup E$. Show that f is measurable iff its restrictions to D and E are measurable. Denote these restrictions by $f|_D$ and $f|_E$.

Hint: Examine the set $\{x \in D \cup E : f(x) > \alpha\} \cap D$.

4) This problem gives some nonmeasurable sets. Let the Borel $\sigma-$algebra on X be $\mathcal{B}(X) = \sigma(\mathcal{C}_O)$ where $\mathcal{C}_O = \mathcal{C}_O(X)$ is the class of open sets that are subsets of X. Then L. measurability on X is very similar to that on $\mathbb{R}$, but $\mathcal{B}(X)$ is contained in the $\sigma-$algebra of measurable sets on X. Let $X = [0, 1)$. Then $m(X) = 1$, and if $A_1, A_2, ...$ are disjoint measurable sets (on X), then $m(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$ where the sum is between 0 and 1 and $0 \leq m(A_i) \leq 1$. Another desirable property for a measure is that $m(E) = m(F)$ if E is congruent to F: E can be transformed into F by translations, rotations and reflections.

Define an equivalence relation on [0,1) by declaring $x \sim y$ iff $x - y$ is rational. Use the Axiom of Choice to find a set $N \subseteq [0, 1)$ which contains precisely one member from each equivalence class. Let $R = \mathbb{Q} \cap [0, 1)$. For each $r \in R$, let
$N_r = \{x + r : x \in N \cap [0, 1 - r)\} \cup \{x + r - 1 : x \in N \cap [1 - r, 1)\}$. The set N_r is obtained by moving the set N r units to the right. Then move the part which sticks out beyond [0,1) one unit to the left. Then N_r is congruent to N and we would like $m(N) = m(N_r)$. The sets N_r are disjoint and each $x \in [0, 1)$ belongs to exactly one N_r. To see this claim, let y be the element of N which belongs to the equivalence class of x. Thus $x \in N_r$ where $r = x - y$ if $x \geq y$ or $r = x - y + 1$ if $x < y$. If $x \in N_r \cap N_s$ with $r \neq s$, then either $x - r$ or $x - r + 1$ and $x - s$ or $x - s + 1$ are distinct elements of N belonging to the same equivalence class, which is impossible since N contains precisely one member of each equivalence class.

Suppose N and N_r are measurable sets on $X = [0, 1)$, and that $m(N) = m(N_r)$ since N and N_r are congruent.

a) Find $m(N \cap [0, 1 - r)) + m(N \cap [1 - r, 1))$ for any $r \in R$.

b) Since $[0, 1) = \cup_{r \in R} N_r$, it follows that $1 = m([0, 1) = \sum_{r \in R} m(N_r)$.

To see that there is a contradiction, note that $m(N_r) = m(N)$ and R is countably infinite.

i) What is $\sum_{r \in R} m(N_r)$ if $m(N) = 0$?

ii) What is $\sum_{r \in R} m(N_r)$ if $m(N) > 0$?

(The contradiction means that N and N_r are not measurable.)