Math 501 HW 8 Spring 2025. Due April 4. Exam2 Friday, March 28.
Final Tuesday, May 6, 8-10AM.

Exam 2 and 3 reviews may be useful.

1) (R. #6, p.89): Let f_n be a sequence of nonnegative functions that converge to f and suppose that $f_n \leq f$ for each n, Prove that $\int f = \lim_{n\to\infty} \int f_n$.

(You may interpret $\int f$ as $\int_{\mathbb{R}} f$ or as $\int_E f$.)

2) Similar to (R. #7, p.89): a) Let the sequence of nonnegative measurable functions f_n be defined by $f_n(x) = 1$ for $n \leq x \leq n+1$ and $f_n(x) = 0$ otherwise. Thus $f_n = \chi_{[n,n+1]}$. Use this sequence to show that we may have strict inequality in Fatou's lemma.

b) Define a decreasing sequence of nonnegative measurable functions f_n by $f_n(x) = 1$ for $x \geq n$ and $f_n(x) = 0$ for $x < n$. Hence $f_n(x) \geq f_{n+1}(x)$ and $f_n = \chi_{[n,\infty)}$. Use this sequence to show that the Monotone Convergence Theorem need not hold for a decreasing sequence of functions.

3) (R. #8, p.89) prove the following generalization of Fatou's lemma. If f_n is a sequence of nonnegative measurable functions, then

$$\int \underline{lim}\, f_n \leq \underline{lim} \int f_n.$$

Hint: Since $f_n \leq f_n$, $g_n = \inf\{f_k : k \geq n\} \leq f_n$ for each $n \in \mathbb{N}$. Since f_n is nonnegative and measurable, g_n is also nonnegative and measurable. Hence $\int \inf\{f_k : k \geq n\} \leq \int f_n$ for all $n \in \mathbb{N}$. Thus

$$\underline{lim} \int \inf\{f_k : k \geq n\} \leq \underline{lim} \int f_n. \tag{1}$$

Now

$$g_n = \inf\{f_k : k \geq n\} \uparrow g = \sup_n \inf\{f_k : k \geq n\} = \underline{lim}\, f_n.$$

Use the hint to complete the proof.

4) Suppose

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \cap [0,1] \\ x^3, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

Find $\int_0^1 f(x)dx$ if possible.

5) For a measurable set E and any constant c, find $\int_E c = \int_E c dx$ if possible.

6) Evaluate $\lim_{n\to\infty} \int_0^1 e^{-nx^2} dx$ using the bounded convergence theorem on $E = (0,1]$.