Math 501 HW 9 Spring 2025. Due April 11. Exam 3 Friday, April 25: a change from the original syllabus.
Final Tuesday, May 6, 8-10AM.

Exam 2 and 3 reviews may be useful.

1) Find $\lim_{n\to\infty} \int_0^1 \frac{nx^{1/2}(sin(nx))^5}{1+n^2x^2} dx$ using LDCT.

Hint: Assume $f_n(x) = \frac{nx^{1/2}(sin(nx))^5}{1+n^2x^2}$ is bounded, continuous, and Riemann integrable and thus L. integrable on (0,1). (Note that $f_n(x)$ is continuous on $[0,1]$, hence bounded and continuous on [0,1], hence Riemann integrable on [0,1].)

Now $(1-nx)^2 \geq 0$, so $1 - 2nx + n^2x^2 \geq 0$, so $1 + n^2x^2 \geq 2nx$ for $x \in (0,1)$. Thus $1/(1+n^2x^2) \leq 1/(2nx)$ for $x \in (0,1)$. Thus

$$|f_n(x)| \leq \frac{nx^{1/2}}{1+n^2x^2} \leq \frac{nx^{1/2}}{2nx} = \frac{1}{2\sqrt{x}} = g(x)$$

on (0,1). Now apply LDCT.

2) Recall Prop 4.3: Let f be defined and bounded on a measurable set E with $m(E) < \infty$. Then $f \in \mathcal{L}(E)$ iff $\inf_{simple\ \psi \geq f} \int_E \psi(x)dx = \sup_{simple\ \phi \leq f} \int_E \phi(x)dx = A$. In proving the proposition, let $\epsilon > 0$. Then $\exists \psi_n \geq f$ such that $\int_E \psi_n - A < 1/(2n)$. Similarly, let $\phi_n \leq f$ with $\int_E \phi_n - A < 1/(2n)$.

Define $\psi_* = \inf_n \psi_n$ and $\phi^* = \sup_n \phi_n$. Let $B = \{x \in E : \phi^*(x) < \psi_*(x)\}$.

Claim: If $m(B) = 0$, then $m(\{x \in E : f(x) \neq \phi^*(x)\}) = 0$.

Prove the claim.

3) Almost (R. #10a, p. 93): Show that if f is integrable over E then i) so is $|f|$, and ii) $|\int_E f| \leq \int_E |f|$.

iii) If f is measurable, then f is integrable iff $|f|$ is integrable.

Let $V \subseteq [0,1] = E$ be a nonmeasurable set. Let

$$f(x) = \begin{cases} 1, & x \in V \\ -1, & x \notin V \end{cases}.$$

Then f, f^+, and f^- are not measurable functions on E, and thus not integrable. Show that $|f|$ is integrable on E.