added 2 to each

Math 501 Exam 1 Spring 2025 Name____________________

Q

YOU ARE BEING GRADED FOR WORK.

1)a) Give the definition of an $\sigma-$algebra on X. is a nonempty class of subsets of X such that i) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and ii) if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.

b) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be $\sigma-$algebras on X. Prove that $\mathcal{F} = \bigcap_{i=1}^{\infty} \mathcal{F}_i$ is a $\sigma-$algebra (on X).

0) $X \in \mathcal{F}_i \ \forall i \ \therefore X \in \mathcal{F}$

i) Let $A_1, A_2, \ldots \in \mathcal{F}$. Then $A_1, A_2, \ldots \in \mathcal{F}_i \ \forall i$

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_i \ \forall i$

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$

ii) Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_i \ \forall i$,

$\therefore A^c \in \mathcal{F}_i \ \forall i$

$\therefore A^c \in \mathcal{F}.$

$\square$

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2) i) Let $D = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c$. Find D^c.

$$= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

ii) Let $B = \cap_{i_1=1}^{\infty} \cup_{i_2=i_1}^{\infty} \cdots \cap_{i_k=i_{k-1}}^{\infty} A_{i_k}^c$. Find B^c.

$$= \bigcup_{i_1=1}^{\infty} \bigcap_{i_2=i_1}^{\infty} \cdots \bigcup_{i_k=i_{k-1}}^{\infty} A_{i_k}$$

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3) Simplify the following set where $a < b$. $B = \cup_{n=1}^{\infty}([-n, a] \cup [b, n])$.

$$= (-\infty, a] \cup [b, \infty)$$

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X = universal subset

4) Let $f : X \to Y$.
Prove $f^{-1}[B^c] = [f^{-1}[B]]^c$ for $B \subseteq Y$.

Let $B \subseteq Y$

LHS $\subseteq$ RHS: Let $x \in f^{-1}[B^c]$.

Then $x \in X$ and $f(x) \in B^c$. $\therefore f(x) \notin B$.

$\therefore x \notin \{x \in X : f(x) \in B\} = f^{-1}[B]$.

$\therefore x \in [f^{-1}[B]]^c$ and $f^{-1}[B^c] \subseteq [f^{-1}[B]]^c$.

RHS $\subseteq$ LHS: Let $x \in (f^{-1}[B])^c$ $(\subseteq X)$.

$\therefore x \notin \{x \in X : f(x) \in B\} = f^{-1}[B]$.

$\therefore f(x) \notin B \Rightarrow f(x) \in B^c$.

$\therefore x \in X$ and $f(x) \in B^c$.

$\therefore x \in f^{-1}[B^c]$

$\therefore (f^{-1}[B])^c \subseteq f^{-1}[B]$

$\square$

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5) Let $X = \mathbb{R}$.

a) By definition, the Borel σ−algebra $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C})$ for some class $\mathcal{C}$ of subsets of $\mathbb{R}$. What is $\mathcal{C}$?

= class of all open subsets of $\mathbb{R}$

b) Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_C$ be the class of all closed ~~intervals~~ sets in $\mathbb{R}$. Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_C)$.

i) $\sigma(\mathcal{C}_c) \subseteq \sigma(\mathcal{C}_o)$:

Let $A \in \mathcal{C}_c$. Then A is a closed set.

$\therefore A^c$ is open $\Rightarrow A^c \in \sigma(\mathcal{C}_o)$.

$\therefore A = (A^c)^c \in \sigma(\mathcal{C}_o)$.

$\therefore \mathcal{C}_c \subseteq \sigma(\mathcal{C}_o)$

$\therefore \sigma(\mathcal{C}_c) \subseteq \sigma(\mathcal{C}_o)$.

ii) $\sigma(\mathcal{C}_o) \subseteq \sigma(\mathcal{C}_c)$:

Let $O \in \mathcal{C}_o$. $\therefore O^c$ is closed.

$\therefore O^c \in \sigma(\mathcal{C}_c)$. $\therefore O = (O^c)^c \in \sigma(\mathcal{C}_c)$

$\therefore \mathcal{C}_o \subseteq \sigma(\mathcal{C}_c)$

$\therefore \sigma(\mathcal{C}_o) \subseteq \sigma(\mathcal{C}_c)$

$\square$

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