add 5 to each

Math 501 Exam 2 Spring 2025 Name____________________

YOU ARE BEING GRADED FOR WORK

1) Let $A, B \in \mathcal{F}_M$, $m(A) = 4$, and $m(A \cap B) = 2$. Let $\phi = 5\chi_A$. Let $\int f = 1$.
For a) – d) calculate

a) $\int \chi_A$ $= m(A) = \boxed{4}$

b) $\int \phi$ $= 5\, m(A) = 5(4) = \boxed{20}$

c) $\int (2f + 3\phi)$ $= 2\int f + 3\int \phi = 2(1) + 3(20) = \boxed{62}$

d) $m(A - B)$. $A = (A \cap B) \cup (A - B)$ disjoint

so $m(A-B) = m(A) - m(A \cap B) = 4 - 2 = \boxed{2}$

e) Show $m(B) \geq 2$. $A \cap B \subseteq B$

$\therefore\ 2 = m(A \cap B) \leq m(B)$

$\overline{}$ 20

2) Let $f_i \in \mathcal{L}(D)$ be measurable functions with domain D.
a) Prove $sup_n\ f_n \in \mathcal{L}(D)$.

$\forall t$, $\{x \in D : \sup_n f_n(x) \leq t\} =$

$\bigcap_{n=1}^{\infty} \{x \in D : f_n(x) \leq t\} \in \mathfrak{F}_M$

$\bigcap_{n=1}^{n \leq \infty}$

b) Prove $inf_n\ f_n \in \mathcal{L}(D)$.

$\forall t$, $\{x \in D : \inf_n f_n(x) \geq t\}$

$= \bigcap_{n=1}^{\infty} \{x \in D : f_n(x) \geq t\} \in \mathfrak{F}_M$

all

ok

at least one

Or $\{x \in D : \inf f_n(x) \leq t\} = \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) \leq t\}$

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Q with fewer hints

3) Let $E \subseteq \mathbb{R}$ be a (Lebesgue) measurable set. Let the number $c \in \mathbb{R}$. Define the set $B = E + c = \{c + x : x \in E\}$. Then B is a translate of E. If $I = (a, b)$ is an open interval, then $I + c = (a + c, b + c)$ and $l(I) = m^*(I) = m^*(I + c) = l(I + c) = b - a$. Let I_n be a sequence of open intervals such that $E \subseteq \cup_{i=1}^{\infty} I_n$. Then $\cancel{E} + c \subseteq \cup_{i=1}^{\infty}(I_i + c)$.

a) Prove that $m^*(E + c) \leq \sum_{i=1}^{\infty} m^*(I_i)$.

$$m^*(E+c) \underset{\text{monotonicity}}{\leq} m^*\left(\bigcup_{i=1}^{\infty}(I_i+c)\right) \underset{\text{subadditivity}}{\leq} \sum_{i=1}^{\infty} m^*(I_i+c) = \sum_{i=1}^{\infty} m^*(I_i).$$

b) The result in a) means that $m^*(E+c)$ is a lower bound for the set defining $m^*(E)$. Thus $m^*(E + c) \leq m^*(E)$ for all $c \in \mathbb{R}$. Applying this result to the set $E - c$ gives $m^*(E) \leq m^*(E - c)$ for all $c \in \mathbb{R}$. Replace c by $-c$ to get $m^*(E) \leq m^*(E + c)$ for all $c \in \mathbb{R}$. Use this result to prove that $m^*(E) = m^*(E + c)$.

(Thus the outer measure m^* and Lebesgue measure m are translation invariant.)

$$m^*(E+c) \leq m^*(E) \leq m^*(E+c)$$

$$\therefore \quad m^*(E) = m^*(E+c)$$

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42) in the notes proof said know

4) a) Let $D = \mathbb{R}$. Then $f : \mathbb{R} \to Y$ is a measurable function ($f \in \mathcal{L}(\mathbb{R})$) if for each $t \in \mathbb{R}$, $f^{-1}[(t, \infty)] = \{x : f(x) > t\} \in \mathcal{F}_M$. Assume that f is a measurable function if for each $t \in \mathbb{R}$, $f^{-1}[(-\infty, t)] = \{x : f(x) < t\} \in \mathcal{F}_M$. Prove that $f \in \mathcal{L}(\mathbb{R})$ if $f^{-1}[[t, \infty)] = \{x : f(x) \geq t\} \in \mathcal{F}_M$ for each $t \in \mathbb{R}$. A B

Suppose $\{x : f(x) \geq t\} \in \mathcal{F}_M$ $\forall t$.

or -1

Then $\{x : f(x) \geq t\}^c = \{x : f(x) < t\} \in \mathcal{F}_M$ $\forall t$.

want to show B $\Rightarrow$ A

Show A $\Rightarrow$ B -3

b) Now suppose that D is not necessarily equal to $\mathbb{R}$ and that $Y = \mathbb{R}^*$ is possible so that f is defined on the extended real numbers. Then $f : D \to Y$ is a measurable function ($f \in \overline{\mathcal{L}}(D)$) if for each $t \in \mathbb{R}$, $f^{-1}[(t, \infty)] = \{x \in D : f(x) > t\} \in \mathcal{F}_M$. Prove that $f \in \overline{\mathcal{L}}(D)$ if $f^{-1}[(-\infty, t]] = \{x \in D : f(x) \leq t\} \in \mathcal{F}_M$ for each $t \in \mathbb{R}$. A B

Suppose $\{x \in D : f(x) \leq t\} \in \mathcal{F}_M$ $\forall t$.

Then $D - \{x \in D : f(x) \leq t\} =$

$\{x \in D : f(x) > t\} \in \mathcal{F}_M$ $\forall t$.

complements! -3

($D \in \mathcal{F}_M$ by def of measurable function)

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want to show B $\Rightarrow$ A

notes 47) said know proof

5) The following two statements are false. Give a counterexample or other justification to prove that the statements are false.

a) If f is a bounded function in [0,1], then f is a measurable function in [0,1].

Let $A \subseteq [0,1]$ be a non measurable set.

Then χ_A is a bounded non measurable function.

to usually

b) If O is an open set in $\mathbb{R}$, then $m(O)$ is positive. ($m(O) = \infty$ including)

$m(\phi) = 0$ and ϕ is an open set.

accept $m^*(O) = \infty$ for $O = \mathbb{R}$, $(0, \infty)$ etc

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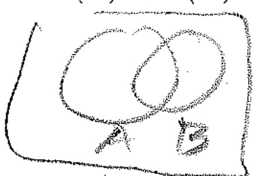
6) a) Suppose A and B are measurable sets, $m(B - A) = 0$, and $m(A) - m(B)$ is not of the form $\infty - \infty$. Find a useful formula for $m(A - B)$.



$A = (A-B) \cup (A \cap B)$ } disjoint unions

$B = (B-A) \cup (A \cap B)$

$\therefore$ $m(A) = m(A-B) + m(A \cup B)$ so $m(A-B) = m(A \cup B) - m(A)$ fine

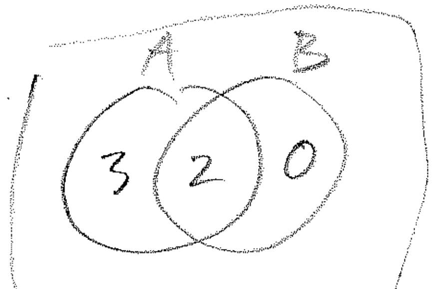
$m(B) = \underbrace{m(B-A)}_{0} + m(A \cap B) = m(A \cap B)$

$\therefore$ ~~$m(A) = m(A-B) + m(B)$~~ so $m(A-B) = m(A) - m(B)$ fine

$m(A-B) = m(A) - m(A \cap B)$ fine

b) Under the conditions of a), assume $m(B-A) = 0$, $m(A \cup B) = 5$, and $m(A-B) = 3$. Find $m(B)$.

$\boxed{m(B) = 2} = m(A \cap B)$



$m(A) = 5 = m(A \cup B)$

$m(B) = m(A) - m(A-B) = 5-3$

or $A \cup B = B \cup (A-B)$ disjoint

$m(B) = m(A \cup B) - m(A-B) = 5-3$

or $m(B) = m(A \cap B) + m(B-A) = 2+0$

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