added 2 to every score   did not prepare them as good enough.

Math 501 Exam 3 Spring 2025   Name____________________

YOU ARE BEING GRADED FOR WORK

1) Suppose functions f_n all have domain D=[0,1] for $n = 1, 2, ...$ Let the measure be L. measure dm. Let

$$f_n(x) = \begin{cases} 0, & 1/n < x \le 1 \\ n, & 0 \le x \le 1/n. \end{cases}$$

a) Find extended real valued function f such that $f_n(x) \to f(x)$ everywhere: so for all $x \in [0, 1]$. (Check $f_n(0)$ and $f_n(1)$ carefully. Note that $f(x) = \pm\infty$ is possible.)

$$f_n(x) \to f(x) = \begin{cases} 0 & x \in (0,1] \\ \infty & x = 0 \end{cases}$$

b) Compute $\int_D f_n \, dm$. $= \int_0^{\frac{1}{n}} n \, dx = n x \Big|_0^{\frac{1}{n}} = n \frac{1}{n} = \boxed{1}$

c) Compute $\lim_{n \to \infty} \int_D f_n \, dm$. $= \lim_{n \to \infty} 1 = 1$

d) Compute $\int_D f \, dm$. $= \int_0^1 0 \, dx = 0$

since $f(x) = 0$ ae on $[0,1]$.

F 2004 Qual 1 a)   (like Quiz 8b)

2) Let function f be L. integrable in [a,b]. Let $F(x) = \int_a^x f(t)dt$ for all $x \in [a,b]$. Show that f is of bounded variation.

F is absolutely continuous. ∴ f is of BV

or Let $a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a,b]$,

Then $* = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \le$

$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt = \int_a^b |f(t)|\,dt < \infty.$

$\therefore T = \sup_{\pi} * = \int_a^b |f(t)|\,dt < \infty$

∴ f is of BV

10

3) Assume f is L. integrable. Prove that if $m(E) = 0$, then $\int_E f = 0$. Here the integral is the L. integral, and you may use results from L. integrals for bounded functions and for nonnegative functions in your proof.

$\int_E f = \int_E f^+ - \int_E f^- = 0 - 0$

by results from L. integrals for nonnegative functions

10

4) a) State Lebesgue's (Dominated) Convergence Theorem.

Let g be integrable over $E \in \mathfrak{F}_m$ and let f_n be a sequence of measurable functions over E such that $|f_n| \le g$ on E and $\lim_{n\to\infty} f_n(x) = f(x)$ ae on E. Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

b) Using the first half of the proof of the LDCT, let $h_n(x) = g(x) - f_n(x)$. Then $h_n \geq 0$ and $h_n \to g - f$ ae on E. Since $|f| \leq g$, f is integrable. Apply Fatou's lemma on $\int_E (g - f)$ to prove that

$$\overline{\lim} \int_E f_n \leq \int_E f.$$

$$\int_E (g-f) \le \underline{\lim} \int_E \underbrace{(g-f_n)}_{h_n}$$

$$\therefore \int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n$$

$$\underline{\lim}(-a_n) = -\overline{\lim}(a_n)$$

Thus $\overline{\lim} \int_E f_n \le \int_E f$

$(\text{no } \infty - \infty)$

20

5) Prove the following theorem. Theorem: A measurable function f is L. integrable over a measurable set $E \in \mathcal{F}_M$ iff $|f|$ is L. integrable over E.

Suppose f is L. integrable.

Then f^+ and f^- are L. integrable,

and $\int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$

$\therefore$ $|f|$ is L. integrable

Suppose $|f|$ is L. integrable,

then $\int_E |f| = \int_E f^+ + \int_E f^- < \infty$

and both integrals need to be finite since both integrals are nonnegative

Hence f^+ and f^- are L. integrable,

$\therefore$ f is L. integrable.

better!

$\int_E f = \int_E f^+ - \int_E f^- \leq \int_E f^+ + \int_E f^- = \int_E |f| < \infty$

$\int_E f^- \geq 0$

10

6) Th. Let $u_i \geq 0$ be L. measurable functions over $E \in \mathcal{F}_M$. Let $f(x) = \sum_{i=1}^{\infty} u_i(x)$ for $x \in E$. Then $\int_E f =$

$$\int_E \sum_{i=1}^{\infty} u_i(x)dx = \sum_{i=1}^{\infty} \int_E u_i(x)dx.$$

Prove the above theorem using $f_n(x) = \sum_{i=1}^{n} u_i(x)$, $f_n \geq 0$, $f_n \uparrow f$ and f_n are L. measurable.

By the MCT, $\lim_{n\to\infty} \int_E f_n = \int_E f$

and $\lim_{n\to\infty} \int_E f_n\,dx = \lim_{n\to\infty} \int_E \sum_{i=1}^{n} u_i(x)\,dx$

$\underset{a}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i(x)\,dx = \sum_{i=1}^{\infty} \int_E u_i(x)\,dx$

a: restricted linearity for finite sum nonneg functions

$\int_E u_i(x)\,dx$ is monotone increasing and has limit $= \sum_{i=1}^{\infty} \int_E u_i(x)\,dx$

MCT or at least -5

10