add 20 to each score

about −12 for very wrong or incomplete proof

Math 501   Final Spring 2025   Name____________________

YOU ARE BEING GRADED FOR WORK

For Qual

1) Suppose functions f_n all have domain D=[0,1] for $n = 1, 2, \ldots$. Let the measure be L. measure dm. Let

$$f_n(x) = \begin{cases} 0, & 1/n^2 < x \le 1 \\ n, & 0 \le x \le 1/n^2. \end{cases}$$

a) Find extended real valued function f such that $f_n(x) \to f(x)$ everywhere: so for all $x \in [0,1]$. (Check $f_n(0)$ and $f_n(1)$ carefully. Note that $f(x) = \pm\infty$ is possible.)

$$\boxed{f_n(x) \to \begin{cases} \infty & x = 0 \\ 0 & x \in (0,1] \end{cases}}$$

b) Compute $\int_D f_n \, dm$. $= \int_0^{\frac{1}{n^2}} n \, dx = nx \Big|_0^{\frac{1}{n^2}} = n \frac{1}{n^2} = \boxed{\frac{1}{n}}$

c) Compute $\lim_{n\to\infty} \int_D f_n \, dm$. $= \lim_{n\to\infty} \frac{1}{n} = \boxed{0}$

d) Compute $\int_D f \, dm$. $\int_0^1 0 \, dx = \boxed{0}$

$f = 0$ ae

70 (17.5 each)

2) a) State Lebesgue's (Dominated) Convergence Theorem.

Let g be integrable over $E \in \mathfrak{M}$ and let f_n be a sequence of measurable functions over E such that $|f_n| \leq g$ on E and $\lim_{n\to\infty} f_n(x) = f(x)$ ae on E.

Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

b) Using the second half of the proof of the LDCT, let $k_n(x) = g(x) + f_n(x)$. Then $k_n \geq 0$ and $k_n \to g + f$ ae on E. Since $|f| \leq g$, f is integrable. Apply Fatou's lemma on $\int_E (g + f)$ to prove that

$$\int_E f \leq \overline{\lim} \int_E f_n.$$

$\int_E (g+f) \leq \underline{\lim} \int_E (g+f_n)$ or

$\int_E g + \int_E f \leq \int_E g + \underline{\lim} \int_E f_n$

$\therefore \int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n$

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3) Let $\mathcal{A}$ be a class of subsets of X. The σ−algebra generated by $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is the intersection of all σ−algebras containing $\mathcal{A}$. Thus $\sigma(\mathcal{A}) = \cap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ where Λ is the collection of σ−algebras $\mathcal{F}_\lambda$ that contain $\mathcal{A}$. Prove that $\sigma(\mathcal{A})$ is a σ−algebra.

-1) Λ is nonempty since the σ- algebra of all subsets of X is in Λ. 3 of 3 -3

0) Since $X \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$, $X \in \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ so $\sigma(\mathcal{A})$ is nonempty.

i) If $A_1, A_2, \ldots \in \sigma(\mathcal{A})$ then $A_1, A_2, \ldots \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$.

$\therefore \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$. $\therefore \bigcup_{i=1}^{\infty} A_i \in \sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$.

ii) If $A \in \sigma(\mathcal{A})$, then $A \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$.

$\therefore A^c \in \mathcal{F}_\lambda \ \forall \lambda \in \Lambda$.

$\therefore A^c \in \sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$.

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4) Let J be any nonempty interval with endpoints $a < b$ where $a = -\infty$ and $b = \infty$ are possible. Prove that the nonempty interval J is uncountable.

If J was countable, then $m(J) = 0$,

but $m(J) = b - a > 0$.

$\therefore J$ is uncountable

-7 if picks like $S \times (a, \infty)$

works for $[0,1]$ not J

assume countable with list

$a_1 = 0.a_{11}\, a_{12}\, a_{13} \ldots$

$a_2 = 0.a_{21}\, a_{22}\, a_{23} \ldots$

$a_3 = 0.a_{31}\, a_{32}\, a_{33} \ldots$

Let $b_j = 0.b_{11}\, b_{12}\, b_{13} \ldots$ $\quad b_{ij} = 1 - a_{ij}$

$\Rightarrow \Leftarrow$

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5) Let $c \in \mathbb{R}$, $f \in \mathcal{L}(D)$, $g \in \mathcal{L}(D)$, and $f_i \in \mathcal{L}(D)$ be measurable functions with domain D. you may assume $a f \in \mathcal{L}(D)$ $\forall a \in \mathbb{R}$.

a) Prove $\max(f, g) \in \mathcal{L}(D)$.

$\forall t$, $\{x \in D : \max(f(x), g(x)) \leq t\} =$

$\{x \in D : f(x) \leq t\} \cap \{x \in D : g(x) \leq t\} \in \mathcal{F}_M$

$(\max(f(x), g(x)) \leq t$ iff both $f(x) \leq t$ and $g(x) \leq t)$

or $\{x : D : \max(f(x), g(x)) > t\}$

(at least one)

$= \{x \in D : f(x) > t\} \cup \{x \in D : g(x) > t\}$

b) Prove $|f| \in \mathcal{L}(D)$.

$|f| = \max(f, -f) \in \mathcal{L}(D)$

or $|f| = f^+ + f^-$, $f^+ = \max\{f, 0\} \in \mathcal{L}(D)$

$f^- = \max\{-f, 0\} \in \mathcal{L}(D)$

by a)
$\forall t$ $\{x \in D : |f(x)| \leq t\} = \{x \in D : f^+(x) \leq t\} \cap \{x \in D : f^- \leq t\}$

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6) The function $f : D \to Y$ is a measurable function ($f \in \mathcal{L}(D)$) if for each $t \in \mathbb{R}$, $f^{-1}[(t, \infty)] = \{x \in D : f(x) > t\} \in \mathcal{F}_M$.

a) Prove that $f \in \mathcal{L}(\mathbb{R})$ if $f^{-1}[[t, \infty)] = \{x \in D : f(x) \geq t\} \in \mathcal{F}_M$ for each $t \in \mathbb{R}$.

Suppose $\{x \in D : f(x) \geq t\} \in \mathcal{F}_M \ \forall t$.

Then $\bigcup_{n=1}^{\infty} \underbrace{\{x \in D : f(x) \geq t + \tfrac{1}{n}\}}_{\in \mathcal{F}_M \text{ for each } n} = \{x \in D : f(x) > t\} \in \mathcal{F}_M$

$\{x \in D : f(x) \geq t\}^c = \{x \in D : f(x) < t\}$ -6
true if $D = \mathbb{R}$

b) Prove that $f \in \mathcal{L}(D)$ if $f^{-1}[(-\infty, t]] = \{x \in D : f(x) \leq t\} \in \mathcal{F}_M$ for each $t \in \mathbb{R}$.

Suppose $\{x \in D : f(x) \leq t\} \in \mathcal{F}_M \ \forall t$

Then $D - \{x \in D : f(x) \leq t\} =$

$\{x \in D : f(x) > t\} \in \mathcal{F}_M \ \forall t$

$\{x \in D : f(x) \leq t\}^c = \{x \in D : f(x) > t\}$ -6
true if $D = \mathbb{R}$

($D \in \mathcal{F}_M$ by def of measurable function.)

or $\{x \in D : f(x) \leq t\} = D - \{x \in D : f(x) > t\}$

$= D - \{x \in D : f(x) > t\}^c$

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7) Unless otherwise stated, assume $f(x)$, $g(x)$, and $f_i(x)$ are measurable and Lebesgue integrable, and that all indicated sets are measurable.

a) Using linearity, if $E = \cup_{i=1}^n E_i$ where the E_i are disjoint, then prove that $\int_E f(x)dx = \sum_{i=1}^n \int_{E_i} f(x)dx$.

$$\int_E f = \int f \chi_{\cup_{i=1} E_i} = \int \sum_{i=1}^n f\chi_{E_i} = \sum_{i=1}^n \int f \chi_{E_i}$$

$$= \sum_{i=1}^n \int_{E_i} f$$

b) If $f(x) \le g(x)$ almost everywhere on E, then prove monotonicity: $\int_E f(x)dx \le \int_E g(x)dx$. You may use results for nonnegative functions.

If $f \le g$ a.e. then $f = f^+ - f^- \le g^+ - g^-$ a.e.

$\therefore f^+ \le g^+$ a.e. and $-f^- \le -g^-$ a.e. or $f^- \ge g^-$ a.e.

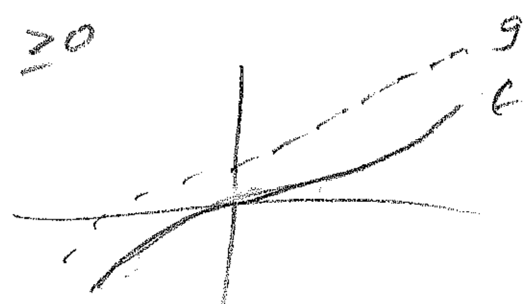
$$\therefore \int_E f = \int_E f^+ - \int_E f^- \le \int_E g^+ - \int_E g^- = \int_E g$$

$\int_E f^+ \le \int_E g^+$ and $\int_E f^- \ge \int_E g^-$ by results for nonneg fns

so $-\int_E f^- \le -\int_E g^-$

Or $g - f \ge 0$ $\therefore \int_E g - f \ge 0$

so $\int_E g \ge \int_E f$



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8) A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* on $[a,b]$ if given $\epsilon > 0$, $\exists \delta > 0$ such that $\sum_{i=1}^{n} |f(x_i+h_i) - f(x_i)| < \epsilon$ for every finite collection $\{(x_i, x_i+h_i)\}$ of nonoverlapping intervals with $\sum_{i=1}^{n} |h_i| < \delta$. ~~where the h_i > 0~~

a) Take $n = 1$ to show that an absolutely continuous function f is continuous on $[a,b]$. (The same proof can be used to show that f is uniformly continuous on $[a,b]$.)

$\forall \varepsilon > 0 \quad \exists \delta > 0 \ni$

$|f(x+h) - f(x)| < \varepsilon \quad$ if $|h| < \delta$

where $x+h, x \in [a,b]$.

b) Let f be L. integrable on $[a,b]$ and let $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$. Let $a = x_0 < x_1 1 < \cdots < x_n = b$ be any partition π of $[a,b]$. Then

$$t = \sum_{i=1}^{n} |F(x_i - F(x_{i-1})| = \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)dt \right|.$$

Use this result to prove that $t \leq c < \infty$ by finding c.
(Hence $T = sup_\pi \ t \leq c < \infty$ and F is of bounded variation.)

$t \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt = \int_a^b |f(t)|\,dt < \infty$

with $c = \int_a^b |f(t)|\,dt$.

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9) Prove that the outer measure m^* is not finitely additive (and thus not countably additive).

Let $V \subseteq X$ be a nonmeasurable set. omitted

Then $\exists A \subseteq X \ni$

or =

$m^*(A) \neq m^*(A \cap V) + m^*(A \cap V^c)$

disjoint

and $A = (A \cap V) \cup (A \cap V^c)$

$\therefore$ finite additivity fails

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