Math 501 Quiz 10 Spring 2025 Name____________________

YOU ARE BEING GRADED FOR WORK

1) Suppose that f is a nondecreasing function. Let $\pi = a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be a partition of $[a,b]$. Then $t = V_a^b(f,\pi) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k}(f(x_i) - f(x_{i-1})) = T$ is a telescoping sum that is equal to T for any partition π. Evaluate the telescoping sum to find T.

$$= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \dots + f(x_k) - f(x_{k-1})$$

$$= f(x_k) - f(x_0) = \boxed{f(b) - f(a)}.$$

34

2) A *measure* μ on a measurable space $(X, \mathcal{F})$ is
i) a nonnegative set function defined for all subsets of $\mathcal{F}$.
ii) $\mu(\emptyset) = 0$
iii) (**countable additivity**): Let $E_1, E_2, ...$ be disjoint measurable sets. Then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Let $a > 0$ and $\nu = a\mu$. Prove that ν is a measure on a measurable space $(X, \mathcal{F})$.

i) ν is a nonnegative set function defined for all subsets of $\mathcal{F}$ since μ is and $a > 0$.

ii) $\nu(\phi) = a\mu(\phi) = a(0) = 0$

iii) Let $E_1, E_2 \dots$ be disjoint measurable sets.

The $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = a\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = a\sum_{i=1}^{\infty}\mu(E_i)$

$$= \sum_{i=1}^{\infty} a\mu(E_i) = \sum_{i=1}^{\infty}\nu(E_i).$$

33

Let f_n be an increasing sequence of nonnegative measurable functions and let $f = \lim f_n$ ae. Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

(14) 3) a) State the Monotone Convergence Theorem.

Let $0 \le f_n \le f_{n+1}$ and f_n measurable on $E \in \mathfrak{M}$. If $f_n \to f$ ae in E, then $\int_E f_n \to \int_E f$.

↑ ok

(19) b) Prove the Monotone Convergence Theorem.

By Fatou's lemma $\int_E f \le \varliminf_{n\to\infty} \int_E f_n$.

so $\int_E f \le \varliminf \int_E f_n \le \varlimsup \int_E f_n \le \int_E f$

$f_n \le f$ ae in E. $\forall n$

(so $\int_E f_n \le \int_E f$ $\forall n$)

33