Math 501 Quiz 11 Spring 2025 Name____________________

YOU ARE BEING GRADED FOR WORK

1) Suppose functions f_n all have domain D=[0,1] for $n = 1, 2, \ldots$. Let the measure be L. measure dm. Let

$$f_n(x) = \begin{cases} 0, & 1/\sqrt{n} < x \le 1 \\ \sqrt{n}, & 0 \le x \le 1/\sqrt{n}. \end{cases}$$

a) Find extended real valued function f such that $f_n(x) \to f(x)$ everywhere: so for all $x \in [0,1]$. (Check $f_n(0)$ and $f_n(1)$ carefully. Note that $f(x) = \pm\infty$ is possible.)

$$f(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \in (0,1] \end{cases}$$

b) Now $f_n - f = f_n = |f_n - f| = |f_n|$ ae in D. Compute $\int_D |f_n - f|\, dm$.

$$= \int_D f_n \, dm = \int_0^{\frac{1}{\sqrt{n}}} \sqrt{n}\, dx = \sqrt{n}\, x \Big|_0^{\frac{1}{\sqrt{n}}} = 1 \quad \forall n$$

c) Compute $\int_D |f_n - f|^2 \, dm = \int_D (f_n - f)^2 \, dm$.

$$= \int f_n^2 \, dm = \int_0^{\frac{1}{\sqrt{n}}} n \, dx = n x \Big|_0^{\frac{1}{\sqrt{n}}} = n \frac{1}{\sqrt{n}} = \sqrt{n}$$

d) Compute $\lim_{n\to\infty} \int_D (f_n - f)^2 \, dm$.

$$= \lim_{n \to \infty} \sqrt{n} = \infty$$

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2) Let $(X, \mathcal{F}, \mu)$ be the measure space, and let f be an integrable function (integrable μ). Let $E \in \mathcal{F}$ be such that $\mu(E) = 0$. If g is a nonnegative integrable function, it can be shown that $\int_E g \, d\mu = 0$.

Use monotonicity and the above result to show that $\int_E f \, d\mu = 0$ since $-|f| \leq f \leq |f|$.

$0 = -\int_E |f| \, d\mu \leq \int_E f \, d\mu \leq \int_E |f| \, d\mu = 0$

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3) Let f be a nonnegative measurable function on $(X, \mathcal{F}, \mu)$. Let $0 < p < \infty$ and $0 < \epsilon < \infty$. Then

$$\int_X [f(x)]^p \, d\mu \geq \int_{\{x \in X : f(x) \geq \epsilon\}} [f(x)]^p \, d\mu \geq \int_{\{x \in X : f(x) \geq \epsilon\}} \epsilon^p \, d\mu.$$

The second inequlaity holds since on the set for the second integral, $[f(x)]^p \geq \epsilon^p$. Evaluate the integral

$$\int_{\{x \in X : f(x) \geq \epsilon\}} \epsilon^p \, d\mu.$$

$= \varepsilon^p \, \mu(\{x \in X : f(x) \geq \varepsilon\})$.

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