Math 501 Quiz 3 Spring 2025 Name____________________

YOU ARE BEING GRADED FOR WORK

1) Consider the sequence $S = \{1/2, 2/3, 3/4, 4/5, ...\} = \{a_n\}$ where $a_n = n/(n+1)$ for $n = 1, 2, 3, ...$

a) Find the least upper bound of $S = \sup_n a_n$.

$= \boxed{1}$

b) Find the greatest lower bound of $S = \inf_n a_n$.

$= \boxed{\frac{1}{2}}$

c) Find $\lim_{n\to\infty} a_n (= \overline{\lim}_n a_n = \underline{\lim}_n a_n)$.

$$= \lim_{n\to\infty} \frac{n}{n+1} \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \boxed{1}$$

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2) Consider the sequence $S = \{\overset{1}{1/2}, \overset{2}{1/3}, \overset{3}{3/4}, \overset{4}{1/5}, ...\} = \{a_n\}$ where $a_n = n/(n+1)$ for n odd $(n = 1, 3, 5,...)$, and $a_n = 1/(n+1)$ for n even $n = 2, 4, ...$

a) Find $\overline{\lim}_n a_n$ = largest limit point of the a_n, where a limit point is also known as a cluster point or accumulation point.

$= \boxed{1}$

$$= \lim_{n\to\infty} \frac{n}{n+1} \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

b) Find $\underline{\lim}_n a_n$ = smallest limit point of the a_n.

$= \boxed{0}$

$$= \lim_{n\to\infty} \frac{1}{n+1}$$

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3) Let $\mathcal{C}$ be the collection of all closed sets of $\mathbb{R}$. Show that $\mathcal{C}$ is not an algebra.

Hint: To show that $\mathcal{C}$ is an algebra, you need to show 0) $\mathcal{C}$ is nonempty. Often this is done by showing that $X \in \mathcal{C}$. i) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$. ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$. So show that i) or ii) fails to hold. many or a unions

$A = [0,1]$, $A^c = (-\infty, 0) \cup (1, \infty)$ is open

$A \in \mathcal{C}$ is closed so A^c is open. $A^c \notin \mathcal{C}$

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4) Let open set $O = (0,5) \cup (3,7) \cup (4,8) \cup (10,100)$.

a) Write O as a disjoint union of two open intervals.

$(0,8) \cup (10,100)$

b) What is the sum of the lengths of these two intervals?
(This sum will turn out to be the Lebesgue measure of O.)

$8 + 90 = 98$

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5) For a proof by contradiction that the distinct binary expansions are not countable, assume that the distinct binary expansions are countable, and listed as follows.

1: $0.a_{11}a_{12}a_{13}a_{14}...$
2: $0.a_{21}a_{22}a_{23}a_{24}...$
3: $0.a_{31}a_{32}a_{33}a_{34}...$
$\vdots$ where $a_{ij} \in \{0,1\}$. Consider the number $c \in [0,1]$ with the binary expansion $0.b_1b_2b_3b_4...$ where $b_i = 1 - a_{ii}$ for $i = 1, 2, 3, ...$

Is the number c in the above list?

no

(diagonal technique)

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