(null sets include $\emptyset$, countable sets and the cantor set)

Math 501   Quiz 4 Spring 2025   Name____________________

YOU ARE BEING GRADED FOR WORK

Qual 1) Let $X = \mathbb{R}$.

(5) a) By definition, the Borel $\sigma-$algebra $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C})$ for some class $\mathcal{C}$ of subsets of $\mathbb{R}$. What is $\mathcal{C}$?

= Class of all open real sets

" open sets $\subseteq \mathbb{R}$ "

(40) b) Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_{OI}$ be the class of all open intervals in $\mathbb{R}$. Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_{OI})$.

i) LHS $\subseteq$ RHS:

Let $O \in \mathcal{C}_O$. Then $O = \bigcup_{i=1}^{\infty} (a_i, b_i)$ (by Prop 3.18).

$\therefore O \in \sigma(\mathcal{C}_{OI})$

$\therefore \mathcal{C}_O \subseteq \sigma(\mathcal{C}_{OI})$

$\therefore \sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_{OI})$

ii) RHS $\subseteq$ LHS:

$\mathcal{C}_{OI} \subseteq \mathcal{C}_O \subseteq \sigma(\mathcal{C}_O)$

$\therefore \mathcal{C}_{OI} \subseteq \sigma(\mathcal{C}_O)$

$\therefore \sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_O)$

$\square$

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2) Give the definition of an open set O of real numbers. if

$\forall x \in O \quad \exists \delta > 0 \ni (x-\delta, x+\delta) \in O$

($\Leftrightarrow \forall x \in O \ \exists \delta > 0 \ni$ each y with $|x-y| < \delta$ belongs to O)

2017 Dual?

3) Prove that [0,1] is uncountable using binary expansions. (Numbers of the form $1/2^i$ have two binary expansions: one that ends in all 0's and one that ends in all 1's. For such a case, use the expansion that ends in all 0's. For example 0.5 has binary expansion $0.10000\ldots = 0.01111\ldots$.)

Assume $[0,1]$ is countable and list all numbers in $[0,1]$ as follows

$a_1 = 0.a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots$

$a_2 = 0.a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots \qquad a_{ij} \in \{0,1\}$

$a_3 = 0.a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots$

$\vdots$

Let $b = 0.b_1\, b_2\, b_3\, b_4 \ldots$

with $b_i = 1 - a_{ii} \in \{0,1\}$.

Then $b \in [0,1]$ and $b \neq a_i$ for $i = 1, 2, \ldots$

Since $b_i \neq a_{ii}$. Contradiction (of the assumption that $[0,1]$ is countable with all numbers listed as the a_i).

$\therefore [0,1]$ is countable.

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