ch. 1 (from Royden):

1.0) Be able to write down the following definitions and theorems.

a) Let $A \subseteq X$ and $B \subseteq X$. Then the **complement of** A is $A^c = \{x \in X : x \notin A\} = \sim A = X - A$. The *difference* $B - A = \{x : x \in B \text{ and } x \notin A\} = B \cap A^c$. The *symmetric difference* $A \triangle B = (A \cap B^c) \cup (B \cap A^c)$.

b) Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq X$.
The **union** $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for at least one $\lambda \in \Lambda\}$.
The **intersection** $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for all $\lambda \in \Lambda\}$.

c) De Morgan's Laws: Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq \Omega$.
i) $[\cup_{\lambda \in \Lambda} A_\lambda]^c = \cap_{\lambda \in \Lambda} A_\lambda^c$. ii) $[\cap_{\lambda \in \Lambda} A_\lambda]^c = \cup_{\lambda \in \Lambda} A_\lambda^c$.

d) Let $f : X \to Y$ be a function where X is the *domain* of f.
The *range* of f is the set $\{y \in Y : \exists x \ni y = f(x)\}$.
The function f is *onto* Y if the range of $f = Y$.
The function f is *one to one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

e) Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$.
i) The **image** under f of A is **the set**
$f[A] = \{y \in Y : y = f(x)$ for some $x \in A\}$.
ii) The **inverse image** of B is **the set**
$f^{-1}[B] = \{x \in X : f(x) \in B\}$.

f) Theorem: Let A_λ be sets for $\lambda \in \Lambda$.
i) $f[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f[A_\lambda]$. ii) $f[\cap_{\lambda \in \Lambda} A_\lambda] \subseteq \cap_{\lambda \in \Lambda} f[A_\lambda]$. iii) $f^{-1}[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f^{-1}[A_\lambda]$.
iv) $f^{-1}[\cap_{\lambda \in \Lambda} A_\lambda] = \cap_{\lambda \in \Lambda} f^{-1}[A_\lambda]$. v) $f^{-1}[A^c] = [f^{-1}[A]]^c$.

g) Let $X \neq \emptyset$. A nonempty class $\mathcal{C}$ of subsets of X is an **algebra** on X (or field) if
i) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$
ii) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$.
An algebra is closed under complements, finite intersections, and finite unions. $X, \emptyset \in \mathcal{C}$. To prove that $\mathcal{C}$ is an algebra, show 0) $\mathcal{C}$ is nonempty, often by showing $X \in \mathcal{C}$. Then show i) and ii).

h) **Know**: Def. Let $X \neq \emptyset$. A nonempty class $\mathcal{F}$ of subsets of X is a $\sigma-$*algebra* on X (or $\sigma-$field) if
i) $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. (To prove that $\mathcal{F}$is nonempty, you can often show that $X \in \mathcal{F}$.)
Note that i) and ii) mean that a σ-algebra is an algebra. A $\sigma-$algebra is closed under countable unions, countable intersections, and complementation. To prove that $\mathcal{F}$ is an $\sigma-$algebra, show 0) $\mathcal{C}$ is nonempty, often by showing $X \in \mathcal{C}$. Then show i) and ii).

I) Let $\mathcal{A}$ be a class of subsets of $X \neq \emptyset$. The $\sigma-$*algebra generated by* $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$ is the intersection of all $\sigma-$algebras containing $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is the smallest $\sigma-$algebra containing $\mathcal{A}$.

j) $\{x_n\}$ is a *Cauchy sequence* if for any $\epsilon > 0$, $\exists N = N_\epsilon \ni \forall n \geq N$ and $\forall m \geq N, |x_n - x_m| < \epsilon$.

k) Th. A sequence $\{x_n\}$ converges (in $\mathbb{R}$) iff $\{x_n\}$ is a Cauchy sequence.

L) A set O is an *open set* of real numbers if $\forall x \in O$, $\exists \delta > 0 \ni (x - \delta, x + \delta) \in O$. Equivalently, for each $x \in O$, $\exists \delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.

m) Th. The intersection of a finite collection of open sets is open.

n) Th. The union of any collection of open sets $\cup_{\lambda \in \Lambda} O_\lambda$ is open.

o) Prop 2.8. Let $O \subseteq \mathbb{R}$ be an open set. Then O is a countable union of disjoint open intervals.
p) A real number x is a *point of closure* of a set E, if $\forall \delta > 0$, $\exists y \in E \ni |x - y| < \delta$. The set of points of closure of E is $\overline{E}$.
q) A set F is a closed set or closed if $F = \overline{F}$.
$\lim x_n = \lim_{n\to\infty} x_n = L \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists N = N_\epsilon \ni \forall n \geq N, |x_n - L| < \epsilon$.
r) Th. A finite union of closed sets is a closed set.
s) Th. The intersection of any collection $\mathcal{C}$ of closed sets is closed.
In other words, let $\mathcal{C} = \{B_\lambda, \lambda \in \Lambda\}$ be closed sets. Then $\cap_{\lambda in \Lambda} B_\lambda$ is a closed set.
t) Th. Let O be and open set and C a closed set. Then O^c is closed and C^c is open.
u) $\emptyset$ and $\mathbb{R}$ are both open and closed.
v) A set E is called a *dense set* in B if $\overline{E} = B$.
w) **Heine Borel Theorem**: Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering.
(In other words, if $F \subseteq \cup_{\lambda \in \Lambda} O_\lambda$, then there is a finite collection $\{O_1, ..., O_n\}$ of sets in Λ $\ni F \subseteq \cup_{i=1}^n O_i$ where n depends on F and Λ.)
x) **Lindelöf Open Covering Theorem**: Let $\mathcal{C} = \{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets of real numbers. Then there is a countable subcollection $\{O_i\}$ of $\mathcal{C}$ such that $\cup_{\lambda \in \Lambda} O_\lambda = \cup_{i=1}^\infty O_i$.
y) The *Cantor set* $C \subseteq [0, 1]$ consists of ternary expansions $0.a_1a_2a_3...$ where $a_i \in \{0, 2\}$ that do not have $a_i = 0$ $\forall i > N$ for some N. It can be shown that the Cantor set C is closed, uncountable, and that there is a 1–1 correspondence between C and [0,1]. Also, the Lebesgue measure $m(C) = 0$.

1.1) Let $\mathcal{A}$ be a class of subsets of X. The $\sigma-$algebra generated by $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$, is the intersection of all $\sigma-$algebras containing $\mathcal{A}$. a) Prove that $\sigma(\mathcal{A})$ is a $\sigma-$algebra.
b) Prove that $\mathcal{A} \subseteq \sigma(\mathcal{A})$.

Proof. a) 0) Let $\sigma(\mathcal{A}) = \cap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ where Λ is the collection of $\sigma-$algebras $\mathcal{F}_\lambda$ that contain $\mathcal{A}$. Then Λ is nonempty since the $\sigma-$algebra of all subsets of X is in Λ. Since $X \in \mathcal{F}_\lambda$ for each λ, it follows that $X \in \sigma(\mathcal{A})$. Thus $\sigma(\mathcal{A})$ is nonempty.

i) If $A_1, A_2, ... \in \sigma(\mathcal{A})$, then $A_1, A_2, ... \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Hence $\cup_{i=1}^\infty A_i \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Thus $\cup_{i=1}^\infty A_i \in \sigma(\mathcal{A})$.

ii) If $A \in \sigma(\mathcal{A})$, then $A \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Hence $A^c \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. Thus $A^c \in \sigma(\mathcal{A})$. $\square$

b) Since $\mathcal{A}, \in \mathcal{F}_\lambda$ for each λ, it follows that $\mathcal{A}, \in \sigma(\mathcal{A})$.

1.2) a) Give the definition of an $\sigma-$algebra on X.

b) Let $\mathcal{F}_1, \mathcal{F}_2, ...$ be $\sigma-$algebras on X. Prove that $\mathcal{F} = \cap_{i=1}^\infty \mathcal{F}_i$ is a $\sigma-$algebra (on X).

1.3) Let $f : X \to Y$ where X is the universal subset.
Prove $f^{-1}[B^c] = \left[f^{-1}[B]\right]^c$ for $B \subseteq Y$.

1.4) Prove that [0,1] is uncountable.

ch. 2

2.1) Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_C$ be the class of all closed real sets. Let $\mathcal{C}_{CI}$ be the class of all closed intervals in $\mathbb{R} = X$. Let $\mathcal{C}_{OI}$ be the class of all open intervals in $\mathbb{R}$. Let $\mathcal{C}_{(a,b]}$ be the class of all half open intervals of the form $(a, b]$ in

$\mathbb{R}$. Let $\mathcal{C}_{[a,b)}$ be the class of all half open intervals of the form $[a, b)$ in $\mathbb{R}$. Let $\mathcal{C}_{[a,b]}$ be the class of all closed intervals of the form $[a, b]$ with $a < b \in \mathbb{R}$. Then the Borel σ-algebra $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_O)$, and it can be shown that $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_C) = \sigma(\mathcal{C}_{CI}) = \sigma(\mathcal{C}_{OI}) = \sigma(\mathcal{C}_{(a,b]}) = \sigma(\mathcal{C}_{[a,b)}) = \sigma(\mathcal{C}_{[a,b]})$.

Proof technique: If $\mathcal{C} \subseteq \sigma$-algebra $\mathcal{F}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

a) Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_C)$.

Proof: i) $(\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_O))$:
Let $A \in \mathcal{C}_C$. Then A is a closed set. Thus A^c is open $\Rightarrow A^c \in \sigma(\mathcal{C}_O)$.
Thus $A = (A^c)^c \in \sigma(\mathcal{C}_O)$.
Thus $\mathcal{C}_C \subseteq \sigma(\mathcal{C}_O)$.
Thus $\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_O)$.

ii) $(\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_C))$:
Let $O \in \mathcal{C}_O$. Then O^c is closed. Thus $O^c \in \sigma(\mathcal{C}_C)$.
Thus $O = (O^c)^c \in \sigma(\mathcal{C}_C)$.
Thus $\mathcal{C}_O \subseteq \sigma(\mathcal{C}_C)$.
Thus $\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_C)$. $\square$

b) Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_{OI})$.

i) $(\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_{OI}))$:
Let $O \in \mathcal{C}_O$. Then $O = \cup_{i=1}^{\infty}(a_i, b_i)$ (by Prop. 2.8).
Thus $O \in \sigma(\mathcal{C}_{OI})$.
Thus $\mathcal{C}_O \subseteq \sigma(\mathcal{C}_{OI})$.
Thus $\sigma(\mathcal{C}_O) \subseteq \sigma(\mathcal{C}_{OI})$.

ii) $(\sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_O))$:
$\mathcal{C}_{OI} \subseteq \mathcal{C}_O$.
Thus $\mathcal{C}_{OI} \subseteq \sigma(\mathcal{C}_O)$.
Thus $\sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_O)$.

c) Prove that $\sigma(\mathcal{C}_C) = \sigma(\mathcal{C}_{CI})$.

i) A closed interval is a closed set. Thus $\mathcal{C}_{CI} \subseteq \mathcal{C}_C \subseteq \sigma(\mathcal{C}_C)$. Hence $\sigma(\mathcal{C}_{CI}) \subseteq \sigma(\mathcal{C}_C)$.

ii) Suppose that A is a closed set. Then A^c is open and $A^c = \sum_{i=1}^{\infty}(a_i, b_i)$. Thus $A = (A^c)^c = \cap_{i=1}^{\infty}(a_i, b_i)^c = \cap_{i=1}^{\infty}[(-\infty, a_i] \cup [b_i, \infty)] \in \sigma(\mathcal{C}_{CI})$ since $(-\infty, a_i]$ and $[b_i, \infty)$ are closed intervals. Thus $\mathcal{C}_C \subseteq \sigma(\mathcal{C}_{CI})$. Hence $\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_{CI})$.

d) Prove that $\sigma(\mathcal{C}_C) = \sigma(\mathcal{C}_{[a,b]})$.

i) A closed interval is a closed set. Thus $\mathcal{C}_{[a,b]} \subseteq \mathcal{C}_C \subseteq \sigma(\mathcal{C}_C)$. Hence $\sigma(\mathcal{C}_{[a,b]}) \subseteq \sigma(\mathcal{C}_C)$.

ii) Suppose that A is a closed set. Then A^c is open and $A^c = \sum_{i=1}^{\infty}(a_i, b_i)$. Thus $A = (A^c)^c = \cap_{i=1}^{\infty}(a_i, b_i)^c = \cap_{i=1}^{\infty}[(-\infty, a_i] \cup [b_i, \infty)] = \cap_{i=1}^{\infty} \cup_{i=1}^{\infty} ([-n, a_i] \cup [b_i, n]) \in \sigma(\mathcal{C}_{[a,b]})$. Thus $\mathcal{C}_C \subseteq \sigma(\mathcal{C}_{[a,b]})$. Hence $\sigma(\mathcal{C}_C) \subseteq \sigma(\mathcal{C}_{[a,b]})$. $\square$

Note: Part a) can be used for $X \neq \mathbb{R}$.

2.2) Let $X = \mathbb{R}$.

a) By definition, the Borel $\sigma-$algebra $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C})$ for some class $\mathcal{C}$ of subsets of $\mathbb{R}$. What is $\mathcal{C}$?

b) Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_C$ be the class of all closed sets in $\mathbb{R}$. Prove that $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_C)$.

ch. 3

3.1) a) Prove $m^*(\emptyset) = 0$.
b) monotonicity: Let $A \subseteq B$. Prove $m^*(a) \le m^*(B)$.
(Let $\mathcal{C}(A) = \{\{I_n\}_{n=1}^N, A \subseteq \cup_{i=1}^N I_n, I_n$ open, $N \le \infty\}$ be the class of open interval covers of A. Let $\mathcal{C}(B) = \{\{J_k\}_{k=1}^M, B \subseteq \cup_{j=1}^M J_k, J_k$ open, $M \le \infty\}$ be the class of open interval covers of A.)
c) Given countable subadditivity, prove finite additivity: $m^*(\cup_{n=1}^N A_n) \le \sum_{n=1}^N m^*(A_n)$.
d) Using $m^*(I) = l(I)$ for any interval I, show that if set E is countable, then $m^*(E) = 0$.
e) Prove the set $E = [0, 1]$ is not countable.

Proof: a) Let $I_n = (-1/n, 1/n)$. then $\emptyset \subseteq I_n \; \forall n \in \mathbb{N}$. Thus $m^*(\emptyset) \le l((-1/n, 1/n)) = 2/n \; \forall n \in \mathbb{N}$. So $0 \le m^*(\emptyset) < \delta$ for any $\delta > 0$. Thus $m^*(\emptyset) = 0$.

b) The result holds if $A = B$, so assume $A \ne B$. Since every open interval cover of B is also a cover of A, $\mathcal{C}(B) \subseteq \mathcal{C}(A)$. Thus $m^*(B) = \inf_{\mathcal{C}(B)} \sum_{k=1}^M l(J_k) \ge \inf_{\mathcal{C}(A)} \sum_{i=1}^N l(I_i) = m^*(A)$ since the inf over the smaller class of covers will be bigger than the inf over the larger class of covers.

c) Take $A_n = \emptyset$ for $n > N$. Then $m^*(\cup_{n=1}^N A_n) = m^*(\cup_{n=1}^\infty A_n) \le \sum_{n=1}^\infty m^*(A_n) = \sum_{n=1}^N m^*(A_n)$.

d) Let $E = \{x_i\}_{i=1}^\infty$. Then $0 \le m^*(E) = m^*(\cup_{i=1}^\infty \{x_i\}) \le \sum_{i=1}^\infty m^*(\{x_i\}) = \sum_{i=1}^\infty m^*([x_i, x_i]) = \sum_{i=1}^\infty 0 = 0$.

e) $m^*([0, 1]) = 1 > 0$.

3.2) a) Prove that E is measurable iff E^c is measurable.
b) Prove $\mathbb{R}$ is measurable.
c) Prove $\emptyset$ is measurable.
d) If $m^*(E) = 0$, prove that E is measurable and $m(E) = 0$.

Proof: a) $m^*(A \cap E^c) + m^*(A \cap (E^c)^c) = m^*(A \cap E) + m^*(A \cap E^c)$.

b) Let $E = \mathbb{R}$. Then $m^*(A \cap \mathbb{R}) + m^*(A \cap \emptyset) = m^*(A) + m^*(\emptyset) = m^*(A)$.

c) proof i): The result follows from a) and b) since $\emptyset = \mathbb{R}^c$.

proof ii): Let $E = \emptyset$. Then $m^*(A \cap \emptyset) + m^*(A \cap \mathbb{R}) = m^*(\emptyset) + m^*(A) = m^*(A)$.

d) Let $A \subseteq \mathbb{R}$. Then $A \cap E \subseteq E$. Thus $m^*(A \cap E) \le m^*(E) = 0$. Similarly, $A \cap E^c \subseteq A$. Thus $m^*(A \cap E^c) \le m^*(A)$. Hence $m^*(A \cap E) + m^*(A \cap E^c) \le m^*(A)$.

3.3) Let $A, B \in \mathcal{F}_M$ with $A \subseteq B$ and $m(A) < \infty$. Prove $m(B - A) = m(B) - m(A)$.

Proof: $B = A \cup (B - A)$ is a disjoint union of $\mathcal{F}_M$ sets. By finite additivity, $m(B) = m(A) + m(B - A)$. Thus $m(B - A) = m(B) - m(A)$ since $m(A) < \infty$.

Proof technique: Write the set of interest as a disjoint union of measurable sets. Then use finite additivity to get the result.

3.4) Th. Let $c \in \mathbb{R}$, $f \in \mathcal{L}(D)$, $g \in \mathcal{L}(D)$, and $f_i \in \mathcal{L}(D)$.
a) $af \in \mathcal{L}(D)$ for any $a \in \mathbb{R}$.
b) $af + bg \in \mathcal{L}(D)$ for any $a, b \in \mathbb{R}$. Hence $\sum_{i=1}^n f_i \in \mathcal{L}(D)$.
c) $\max(f, g) \in \mathcal{L}(D)$. Hence $\max(f_1, ..., f_n) \in \mathcal{L}(D)$.
d) $\min(f, g) \in \mathcal{L}(D)$. Hence $\min(f_1, ..., f_n) \in \mathcal{L}(D)$.
e) $fg \in \mathcal{L}(D)$. Hence $f_1 \cdots f_n \in \mathcal{L}(D)$.
f) $f/g \in \mathcal{L}(D)$ if $g(x) \ne 0 \; \forall \; x \in D$.
g) $sup_n \; f_n \in \mathcal{L}(D)$.
h) $inf_n \; f_n \in \mathcal{L}(D)$.

i) $limsup_n\ f_n \in \mathcal{L}(D)$.
j) $liminf_n\ f_n \in \mathcal{L}(D)$.
k) If $lim_n f_n = f$, then $f \in \mathcal{L}(D)$.
l) $|f| \in \mathcal{L}(D)$.

Proof: a) If $a > 0$, then $\{x \in D : af(x) \leq t\} = \{x \in D : f(x) \leq t/a\} \in \mathcal{F}_M$.
If $a < 0$, then $\{x \in D : af(x) \leq t\} = \{x \in D : f(x) \geq t/a\} \in \mathcal{F}_M$.
If $a = 0$, then $af \equiv 0$ is a constant, and a constant function is a measurable function.
Thus $af \in \mathcal{L}(D)$ if $f \in \mathcal{L}(D)$.

b) For each t,

$$\{x \in D : f(x) + g(x) < t\} = \cup_{r \in \mathbb{Q}} \left[\{x \in D : f(x) < r\} \cap \{x \in D : g(x) < t - r\}\right] \in \mathcal{F}_M$$

since the union is countable. Thus a sum of two measurable functions, each with domain D, is a measurable function, and by induction, a finite sum of measurable functions, each with domain D, is measurable function.

c) $\forall t$, $\{x \in D : \max(f(x), g(x)) \leq t\} = \{x \in D : f(x) \leq t\} \cap \{x \in D : g(x) \leq t\} \in \mathcal{F}_M$ (since $\max(f(x), g(x)) \leq t$ iff both $f(x) \leq t$ and $g(x) \leq t$).

d) $\forall t$, $\{x \in D : \min(f(x), g(x)) \leq t\} = \{x \in D : f(x) \leq t\} \cup \{x \in D : g(x) \leq t\} \in \mathcal{F}_M$ (since $\min(f(x), g(x)) \leq t$ iff at least one of the following holds i) $f(x) \leq t$ or ii) $g(x) \leq t$).

e) First show that $f^2 \in \mathcal{L}(D)$ if $f \in \mathcal{L}(D)$. For any $t \geq 0$, $\{x \in D : f(x)^2 \leq t\} = \{x \in D : -\sqrt{t} \leq f(x) \leq \sqrt{t}\} = \{x \in D : f(x) \leq \sqrt{t}\} - \{x \in D : f(x) < -\sqrt{t}\} \in \mathcal{F}_M$, while for any $t < 0$, $\{x \in D : f(x)^2 \leq t\} = \emptyset \in \mathcal{F}_M$. Thus $f^2 \in \mathcal{L}(D)$. Then $f(x)g(x) = 0.5[(f(x) + g(x))^2 - f(x)^2 - g(x)^2] \in \mathcal{L}(D)$ by b).

f) First show $1/g \in \mathcal{L}(D)$. Then the result follows by e). Now

$$\left\{x \in D : \frac{1}{g(x)} \leq t\right\} = \left\{ \begin{array}{cc} \{x \in D : g(x) \geq 1/t\} \cup \{x \in D : g(x) \leq 0\}, & t \geq 0 \\ \{x \in D : g(x) \geq 1/t\}, & t < 0. \end{array} \right.$$

(Note that for $t = 0$ and $x \in D$, then $1/g(x) \leq 0$ iff $g(x) \leq 0$ since $g(x) \neq 0\ \forall x \in D$.) Hence $LHS \in \mathcal{F}_M$.

g) For each t, $\{x \in D : \sup_n f_n(x) \leq t\} = \cap_{n=1}^{\infty}\{x \in D : f_n(x) \leq t\} \in \mathcal{F}_M$.

h) For each t, $\{x \in D : \inf_n f_n(x) \geq t\} = \cap_{n=1}^{\infty}\{x \in D : f_n(x) \geq t\} \in \mathcal{F}_M$.

i) $limsup_n\ f_n = \inf_k\ \sup_{m \geq k} f_m = \inf_k g_k \in \mathcal{L}(D)$ by h).

j) $liminf_n\ f_n = \sup_k\ \inf_{m \geq k} f_m = \sup_k h_k \in \mathcal{L}(D)$ by g).

k) $f = \limsup_n\ f_n = \liminf_n\ f_n \in \mathcal{L}(D)$ by i) and j).

l) $|f| = \max(f, -f) \in \mathcal{L}(D)$ by c).