Math 501 Exam 1 is Wednesday, Feb. 19, 3:00-3:50 NO NOTES. CHECK FORMULAS: YOU ARE RESPONSIBLE FOR ANY ERRORS ON THIS HANDOUT!

1) The **universal set** X is the set of all elements under consideration. Subsets of X are of interest. The **empty set** is $\emptyset$. Let $A \subseteq X$ and $B \subseteq X$. Then the **complement of** A is $A^c = \{x \in X : x \notin A\} = \sim A = X - A$. The *difference* $B - A = \{x : x \in B$ and $x \notin A\} = B \cap A^c$. The *symmetric difference* $A \triangle B = (A \cap B^c) \cup (B \cap A^c) =$ set of all points that belong to one or the other of both sets but not to both. $[A^c]^c = A$ and $\emptyset = X^c$.

2) **Def.** Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq X$. Then $\{A_\lambda\}_{\lambda \in \Lambda}$ is an indexed family of sets ($= \mathcal{C} = \{A_\lambda, \ni \lambda \in \Lambda\}$, a nonempty collection of sets corresponding to Λ.)

a) The **union** $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for at least one $\lambda \in \Lambda\}$.

b) The **intersection** $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for all $\lambda \in \Lambda\}$.

Notation: a) Often "$\in X$" will be omitted. Hence
$\{x \in X : x \in A_\lambda$ for all $\lambda \in \Lambda\} = \{x : x \in A_\lambda$ for all $\lambda \in \Lambda\}$.

b) Often $\Lambda = \mathbb{N} = \{i\}_{i=1}^\infty = \{1, 2, ...,\} =$ the set of positive integers $=$ the set of *natural numbers*. Then $\bigcup_{\lambda \in \mathbb{N}} A_\lambda = \bigcup_{i=1}^\infty A_i$, and $\bigcap_{\lambda \in \mathbb{N}} A_\lambda = \bigcap_{i=1}^\infty A_i$.

c) If $\Lambda = \{i\}_{i=m}^\infty = \{m, (m+1), ...,\} =$ the set of integers $\geq m$, then
$\bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{i=m}^\infty A_i$, and $\bigcap_{\lambda \in \Lambda} A_\lambda = \bigcap_{i=m}^\infty A_i$.

Warning: Since ∞ is not an integer, there is no set A_∞ in $\bigcup_{i=m}^\infty A_i$ or $\bigcap_{i=m}^\infty A_i$.

3) One way to prove $A = B$ is to prove $A \subseteq B$ and $B \subseteq A$. This technique is equivalent to i) showing that if $x \in A$, then $x \in B$, and ii) showing that if $x \in B$, then $x \in A$. A second way to prove $A = B$ is to show $x \in A$ iff $x \in B$ where "iff" means "if and only if."

4) De Morgan's Laws: Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq \Omega$.
i) $[\cup_{\lambda \in \Lambda} A_\lambda]^c = \cap_{\lambda \in \Lambda} A_\lambda^c$. ii) $[\cap_{\lambda \in \Lambda} A_\lambda]^c = \cup_{\lambda \in \Lambda} A_\lambda^c$. iii) $[\cap_{i=1}^\infty A_i]^c = \cup_{i=1}^\infty A_i^c$.
iv) $[\cup_{i=1}^\infty A_i]^c = \cap_{i=1}^\infty A_i^c$. v) $[A \cup B]^c = A^c \cap B^c$. vi) $[A \cap B]^c = A^c \cup B^c$.

5) Let $f : X \to Y$ be a function where X is the *domain* of f.
The *range* of f is the set $\{y \in Y : \exists x \ni y = f(x)\}$.
The function f is *onto* Y if the range of $f = Y$.
The function f is *one to one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

6) Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$.
i) Def. The **image** under f of A is **the set**
$f[A] = \{y \in Y : y = f(x)$ for some $x \in A\}$.
ii) Def. The **inverse image** of B is **the set**
$f^{-1}[B] = \{x \in X : f(x) \in B\}$.

Warning: $f[A]$ and $f^{-1}[B]$ are sets that depends on the function f. The inverse function need not exist.

7) Theorem: Let A_λ be sets for $\lambda \in \Lambda$. $f : X \to Y$, X the universal subset
a) $f[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f[A_\lambda]$. b) $f[\cap_{\lambda \in \Lambda} A_\lambda] \subseteq \cap_{\lambda \in \Lambda} f[A_\lambda]$. c) $f^{-1}[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f^{-1}[A_\lambda]$.
d) $f^{-1}[\cap_{\lambda \in \Lambda} A_\lambda] = \cap_{\lambda \in \Lambda} f^{-1}[A_\lambda]$. e) $f^{-1}[A^c] = [f^{-1}[A]]^c$.

8) Def. Let $X \neq \emptyset$. A nonempty class $\mathcal{C}$ of subsets of X is an **algebra** on X (or field) if a1) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$
a2) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$.
An algebra is closed under complements, finite intersections, and finite unions. $X, \emptyset \in \mathcal{C}$.

Know: To prove that $\mathcal{C}$ is a algebra, show 0) $\mathcal{C}$ is nonempty. Often this is done by showing that $X \in \mathcal{C}$. Then show i) and ii).

9) **Know**: Def. Let $X \neq \emptyset$. A nonempty class $\mathcal{F}$ of subsets of X is a $\sigma-$*algebra* on X (or $\sigma-$field) if
i) $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.
Note that i) and ii) mean that a σ-algebra is an algebra. A $\sigma-$algebra is closed under countable unions, countable intersections, and complementation. The term "on X" is often understood and omitted. **Common error:** Use n instead of ∞ in i).

Know: To prove that $\mathcal{F}$ is a $\sigma-$algebra, show 0) $\mathcal{F}$ is nonempty. Often this is done by showing that $X \in \mathcal{F}$. Then show i) and ii).

10) **Def.** Let $\mathcal{A}$ be a class of subsets of $X \neq \emptyset$. The $\sigma-$*algebra generated by* $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$ is the intersection of all $\sigma-$algebras containing $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is the smallest $\sigma-$algebra containing $\mathcal{A}$.

Know: Let $\mathcal{F} = \sigma(\mathcal{A}) = \cap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ where Λ is the collection of $\sigma-$algebras $\mathcal{F}_\lambda$ that contain $\mathcal{A}$. To prove that $\sigma(\mathcal{A})$ is a $\sigma-$algebra, first show Λ is nonempty since the $\sigma-$algebra of all subsets of $X = \mathbb{P}(X) \in \Lambda$. Then $A, A_1, A_2, ... \in \mathcal{F}$ implies that $A, A_1, A_2, ... \in \mathcal{F}_\lambda \forall \lambda$. Use this result to show that A^c and $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda \forall \lambda$. Thus A^c and $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

11) **Def.** Let $\mathcal{A}$ be the class of all open intervals (a, b) in [0,1]. Then $\sigma(\mathcal{A}) = \mathcal{B}[0, 1]$ is the Borel $\sigma-$algebra on [0,1].
Fact: $\mathcal{B}[0, 1] = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the class of all closed intervals in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $(a, b]$ in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $[a, b)$ in [0,1].

12) Def. $A_n \uparrow A$ means $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_{i=1}^{\infty} A_i$.
$A_n \downarrow A$ means $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \cap_{i=1}^{\infty} A_i$. See 16).

13) Def. Let A_n be a sequence of sets.
$\overline{lim}\ A_n = limsup_n\ A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \{x : x \in A_n$ for infinitely many $A_n\}$.
$\underline{lim}\ A_n = liminf_n\ A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \{x : x \in A_n$ for all but finitely many $A_n\}$.
If $A_n \in \mathcal{F}$, then $\overline{lim}\ A_n, \underline{lim}\ A_n \in \mathcal{F}$. Also, $liminf_n\ A_n \subseteq limsup_n\ A_n$.

14) Def. If $liminf_n\ A_n = limsup_n\ A_n$, then $lim_n\ A_n = A = liminf_n\ A_n = limsup_n\ A_n$, written $A_n \to A$. (The subscript n is sometimes omitted.)

15) **Theorem.** Let A_n be a sequence of $\mathcal{F}$ sets.
a) $\overline{lim}\ A_n, \underline{lim}\ A_n \in \mathcal{F}$.
b) If $lim_n\ A_n$ exists, then $lim_n\ A_n = A \in \mathcal{F}$.
c) $liminf_n\ A_n \subseteq limsup_n\ A_n$.
d) $(limsup_n\ A_n)^c = liminf_n\ A_n^c$.
e) $(liminf_n\ A_n)^c = limsup_n\ A_n^c$.

Proof. a) $C_n = \cup_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n. Hence $\cap_{n=1}^{\infty} C_n = \overline{lim}\ A_n \in \mathcal{F}$. $B_n = \cap_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n. Hence $\cup_{n=1}^{\infty} B_n = \underline{lim}\ A_n \in \mathcal{F}$.

b) Follows from a).

c) If $x \in A_n$ for all but finitely many A_n, then $x \in A_n$ for all but infinitely many A_n. Hence if $x \in liminf_n\ A_n$ then $\omega \in limsup_n\ A_n$. Thus $liminf_n\ A_n \subseteq limsup_n\ A_n$.

d) By De Morgan's laws applied twice, $(limsup_n\ A_n)^c = [\cap_{n=1}^{\infty} C_n]^c = \cup_{n=1}^{\infty} C_n^c = liminf_n A_n^c$ where C_n is given in a).

e) By De Morgan's laws applied twice, $(liminf_n\ A_n)^c = [\cup_{n=1}^\infty B_n]^c = \cap_{n=1}^\infty B_n^c = limsup_n A_n^c$ where B_n is given in a). $\square$

Remark 1.6. a) If $limsup_n\ A_n \subseteq A \subseteq liminf_n\ A_n$, then $lim_n A_n = A$ by Theorem in point 15).
b) $B_n = \cap_{k=n}^\infty A_k \uparrow \underline{lim}\ A_n$. Thus $lim_{n\to\infty} \cap_{k=n}^\infty A_k = \underline{lim}\ A_n$.
c) $C_n = \cup_{k=n}^\infty A_k \downarrow \overline{lim} A_n$. Thus $lim_{n\to\infty} \cup_{k=n}^\infty A_k = \overline{lim}\ A_n$, and $\overline{lim}\ A_n = \cap_{n=1}^\infty C_n$.
d) **Do not treat convergence of sets like convergence of functions**. $A_n \to A$ iff $limsup_n\ A_n = liminf_n\ A_n$ which implies that if $\omega \in A_n$ for infinitely many n, then $\omega \in A_n$ for all but finitely many n.
e) **Warning:** Students who have not figured out the examples 16) and 17) tend to make errors on similar problems.
f) Typically want to show that open, closed, and half open intervals can be written as a countable union or countable intersection of intervals of another type. Then the Borel σ-field $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$ where $\mathcal{C}$ is a class of intervals such as the class of all open intervals.

16) **Example.** Prove the following results.
a) $A_1 \subseteq A_2 \subseteq \cdots$ implies that $A_n \uparrow A = \cup_{n=1}^\infty A_n$.
b) $A_1 \supseteq A_2 \supseteq \cdots$ implies that $A_n \downarrow A = \cap_{n=1}^\infty A_n$.

Proof. a) For each n, $A = \cup_{k=n}^\infty A_k$. Thus $limsup\ A_n = \cap_{n=1}^\infty A = A$. For each n, $\cap_{k=n}^\infty A_k = A_n$. Thus $liminf\ A_n = \cup_{n=1}^\infty A_n = A$.

b) For each n, $\cup_{k=n}^\infty A_k = A_n$. Thus $limsup\ A_n = \cap_{n=1}^\infty A_n = A$. For each n, $\cap_{k=n}^\infty A_k = A$. Thus $liminf\ A_n = \cup_{n=1}^\infty A = A$.

17) **Example.** Simplify the following sets where $a < b$. Answers might be $(a,b), [a,b), (a,b], [a,b], [a,a] = \{a\}, (a,a) = \emptyset$.

a) $I = \bigcap_{n=1}^\infty \left(a, b + \frac{1}{n}\right)$, b) $I = \bigcup_{n=1}^\infty \left(a, b - \frac{1}{n}\right]$, c) $I = \bigcup_{n=1}^\infty \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$,
d) $I = \bigcap_{n=1}^\infty \left[a, b + \frac{1}{n}\right)$, e) $I = \bigcap_{n=1}^\infty \left[a, a + \frac{1}{n}\right)$, f) $I = \bigcup_{n=1}^\infty \left[a, b - \frac{1}{n}\right]$.

Solution. a) $I = (a,b] = \bigcap_{n=1}^\infty \left(a, b + \frac{1}{n}\right) = \bigcap_{n=1}^\infty A_n$ where $A_n \downarrow I$. Note that $(a,b] \subseteq A = \bigcap_{n=1}^\infty \left(a, b + \frac{1}{n}\right)$ since $b \in \left(a, b + \frac{1}{n}\right)$ $\forall n$. For any $\epsilon > 0$, $(a, b+\epsilon] \not\subseteq A$ since $b + 1/n < b + \epsilon$ for large enough n. Note that $b + 1/n \to b$, but sets are not functions. (A common error is to say $I = (a,b)$.)
b) $I = (a,b) = \bigcup_{n=1}^\infty \left(a, b - \frac{1}{n}\right] = \bigcup_{n=1}^\infty A_n$ where $A_n \uparrow I$. Note that $b \notin \bigcup_{n=1}^\infty \left(a, b - \frac{1}{n}\right] = A$ since $b \notin \left(a, b - \frac{1}{n}\right]$ $\forall n$ and since $n \in \mathbb{N}$ so $n = \infty$ never occurs. Note that $\left(a, b - \frac{1}{n}\right] = \emptyset$ if $b - 1/n \leq a$. For any $\epsilon > 0$ such that $b - \epsilon > a$, it follows that $(a, b-\epsilon] \in A$ since $b - 1/n > b - \epsilon$ for large enough n, say $n > N_\epsilon$. Thus $b - \epsilon \in A$ all but finitely many times.
c) $I = (a,b) = \bigcup_{n=1}^\infty \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=1}^\infty A_n$ where $A_n \uparrow I$. Note that $a, b \notin A = I$ since

$a, b \notin \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$ $\forall n \in \mathbb{N}$. Then the proof is similar to that of b).

d) $I = [a,b] = \bigcap_{n=1}^{\infty} \left[a, b+\frac{1}{n}\right) = \bigcap_{n=1}^{\infty} A_n$ where $A_n \downarrow I$. This proof is similar to that of a).

e) $I = [a,a] = \{a\} = \bigcap_{n=1}^{\infty} \left[a, a+\frac{1}{n}\right) = \bigcap_{n=1}^{\infty} A_n$ where $A_n \downarrow I$. Note that $a \in A = I$, but $a + \epsilon \notin A$ $\forall \epsilon > 0$.

f) $I = [a,b) = \bigcup_{n=1}^{\infty} \left[a, b-\frac{1}{n}\right] = \bigcup_{n=1}^{\infty} A_n$ where $A_n \uparrow I$. This proof is similar to that of b).

Chapter 2:

18) Let $S \subseteq \mathbb{R}$. Then d is an upper bound for S if for each $x \in S$, $x \leq d$. Then $c = sup\ S = sup_{x \in S}\ x$ is the least upper bound for S if $sup\ S$ is an upper bound for S, and if $sup\ S \leq d$ for any upper bound for S. Similarly, a is a lower bound for S if for each $x \in S$, $x \geq a$. Then $b = inf\ S = inf_{x \in S}\ x$ is the greatest lower bound for S if $inf\ S$ is a lower bound for S, and if $inf\ S \geq a$ for any lower bound for S. Both $sup\ S$ and $inf\ S$ are unique when they exist.

Facts: I) $sup\ S = c \in \mathbb{R}$ iff i) $\forall x \in S, x \leq c$, ii) $\forall \epsilon > 0, \exists x_0 \in S \ni c - \epsilon < x_0 \leq c$.
II) $inf\ S = b \in \mathbb{R}$ iff i) $\forall x \in S, x \geq b$, ii) $\forall \epsilon > 0, \exists x_0 \in S \ni b \leq x_0 < b + \epsilon$.

19) **Completeness Axiom**: Every nonempty set S of real numbers which has an upper bound has a least upper bound.

20) $inf_{x \in S}\ x = -sup_{x \in S}\ -x$. Thus any nonempty set S of real numbers which has a lower bound has a greatest lower bound.

21) Let $\{x_n\}_{n=1}^{\infty} = \{x_n\}$ be a sequence of real numbers.
a) $x_n \uparrow x$ means $x_1 \leq x_2 \leq \cdots$ and $x_n \to x$. (nondecreasing sequence)
b) $x_n \downarrow x$ means $x_1 \geq x_2 \geq \cdots$ and $x_n \to x$. (nonincreasing sequence)
(These monotone sequences have "limits" if $\pm\infty$ are allowed. So $x_n \uparrow \infty$ means the sequence diverges to ∞, while $x_n \downarrow -\infty$ means the sequence diverges to $-\infty$. If x is a real number, then x is the limit of the sequence.)

22) The limit superior and limit inferior of a sequence will be useful. The sequence $\{a_n\}_{n=1}^{\infty} = a_1, a_2, ...$ Let $\{a_n\}_{n=m}^{\infty} (= a_m, a_{m+1}, ...)$ be a sequence of numbers. Then
i) $\sup a_n$ = least upper bound of $\{a_n\}$, and
ii) $\inf a_n$ = greatest lower bound of $\{a_n\}$.

23) **Def.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

a) The limit superior of the sequence $limsup_n a_n = \overline{lim}_n a_n = lim_n\ sup_{k \geq n}\ a_k$ is the limit of the nonincreasing sequence $\{\sup_{k \geq j} a_k\}_{j=1}^{\infty}$.

b) The limit inferior of the sequence $liminf_n a_n = \underline{lim}_n a_n = lim_n\ inf_{k \geq n} a_k$ is the limit of the nondecreasing sequence $\{\inf_{k \geq j} a_k\}_{j=1}^{\infty}$.

24) **Remark.** a) Unlike the limit, $\overline{lim}_n a_n$ and $\underline{lim}_n a_n$ always exist when $\pm\infty$ are allowed as limits, since limits of nondecreasing and nonincreasing sequences then exist.

b) $\underline{lim}_n a_n \leq \overline{lim}_n a_n$

c) $lim_n a_n = a$ iff $\underline{lim}_n a_n = \overline{lim}_n a_n = a$. Hence the limit of a sequence exists iff $\underline{lim}_n a_n = \overline{lim}_n a_n$. Again, $a = \pm\infty$ is allowed.

d) Let $lim^*_n a_n$ be $\overline{lim}_n a_n$ or $\underline{lim}_n a_n$.
If $a_n \leq b_n$, then $lim^*_n a_n \leq lim^*_n b_n$.
If $a_n < b_n$, then $lim^*_n a_n \leq lim^*_n b_n$.
If $a_n \geq b_n$, then $lim^*_n a_n \geq lim^*_n b_n$.
If $a_n > b_n$, then $lim^*_n a_n \geq lim^*_n b_n$.
That is, when taking the $liminf$ or $limsup$ on both sides of a strict inequality, the $<$ or $>$ must be replaced by $\leq$ or $\geq$.

A similar result holds for limits if both limits exist.

e) $limsup_n(-a_n) = -liminf_n a_n$.

f) i) $limsup_n a_n = \overline{lim}_n a_n$ is the limit of the nonincreasing sequence

$$\sup_{k \geq m} a_k, \quad \sup_{k \geq m+1} a_k, ...$$

ii) $liminf_n a_n = \underline{lim}_n a_n$ is the limit of the nondecreasing sequence

$$\inf_{k \geq m} a_k, \quad \inf_{k \geq m+1} a_k, ...$$

iii)

$$\overline{lim}_n a_n = \inf_n \sup_{k \geq n} a_k = \lim_{k \to \infty} \sup(a_n, n \geq k).$$

iv)

$$\underline{lim}_n a_n = \sup_n \inf_{k \geq n} a_k = \lim_{k \to \infty} \inf(a_n, n \geq k).$$

v) If a limit point of a sequence $\{a_n\}$ is any number, including $\pm\infty$, that is a limit of some subsequence, then $liminf_n a_n$ and $limsup_n a_n$ are the inf and sup of the set of limit points, often the smallest and largest limit points.

25) $x_0 \in \mathbb{R}$ is a cluster point of $\{x_n\}$ if $\forall \epsilon > 0$ and $\forall N \in \mathbb{N}, \exists n \geq N \ni |x_n - x_0| < \epsilon$. Thus infinitely many terms of the sequence $\{x_n\}$ are within ϵ of x_0.
∞ is a cluster point of $\{x_n\}$ if given $\beta > 0$ and given N, $\exists n \geq N \ni x_n > \beta$. Thus infinitely many terms of the sequence $\{x_n\}$ are $> \beta$.
$-\infty$ is a cluster point of $\{x_n\}$ if given $\alpha < 0$ and given N, $\exists n \geq N \ni x_n < \alpha$. Thus infinitely many terms of the sequence $\{x_n\}$ are $< \alpha$.
A limit point is also called an accumulation point and a cluster point. If $\{x_n\}$ is a bounded sequence, then $\overline{lim}\ x_n =$ largest accumulation point (cluster point) of $\{x_n\}$, and $\underline{lim}\ x_n =$ smallest accumulation point of $\{x_n\}$.
$x_0 \in \mathbb{R}$ is a cluster point of $\{x_n\}$

26) **Warning:** A common error is to take the limit of both sides of an equation $a_n = b_n$ or of an inequality, e.g. $a_n \leq b_n$. Taking the limit is an error if the existence of the limit has not been shown. If $\pm\infty$ are allowed, $\underline{lim}_n a_n$ and $\overline{lim}_n a_n$ always exits. Hence the $\underline{lim}_n a_n$ or $\overline{lim}_n a_n$ of the above equation or inequality can be taken.

27) $\mathbb{R}_E = \mathbb{R}^* = \mathbb{R} \cup \infty \cup -\infty =$ the *set of extended real numbers.* Note that $\infty - \infty$ is not defined, but by convention, $0(\infty) = 0 = 0(-\infty)$. (It is not clear if $\infty/\infty = 0(\infty) = 0$ or if ∞/∞ is undefined.)

28) **Axiom of Archimedes**: Given any real number x, there is an integer n such that $x < n$.

29) Theorem: If $x < y$, then there exists rational $r \in \mathbb{Q}$ such that $x < r < y$.

30) Def. $\lim x_n = \lim_{n\to\infty} x_n = L \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists N = N_\epsilon \ni \forall n \geq N, |x_n - L| < \epsilon$.

31) Def. $\{x_n\}$ is a *Cauchy sequence* if for any $\epsilon > 0$, $\exists N = N_\epsilon \ni \forall n \geq N$ and $\forall m \geq N, |x_n - x_m| < \epsilon$.

32) Th. A sequence $\{x_n\}$ converges (in $\mathbb{R}$) iff $\{x_n\}$ is a Cauchy sequence.

33) $(a, b]$ and $[a, b)$ are called half open intervals.

34) Notation: let O and O_i denote open sets.

35) Fact: $\emptyset$ and $\mathbb{R}$ are both open and closed.

36) The set of all accumulation points of a set E is denoted by E'.

37) Def. A set O is an *open set* of real numbers if $\forall x \in O$, $\exists \delta > 0 \ni (x-\delta, x+\delta) \in O$. Equivalently, for each $x \in O$, $\exists \delta > 0$ such that each y with $|x - y| < \delta$ belongs to O.

38) Th. The intersection of a finite collection of open sets is open.

39) Th. The union of any collection of open sets $\cup_{\lambda\in\Lambda} O_\lambda$ is open.

40) $\cap_{i=1}^{\infty}(0 - 1/n, 0 + 1/n) = \{0\}$ which is not open.
(This quantity is an intersection of a countably infinite collection of open sets, so does not contradict 38).)

41) Th. Let $O \subseteq \mathbb{R}$ be an open set. Then O is a countable union of disjoint open intervals.

42) Def. A real number x is a *point of closure* of a set E, if $\forall \delta > 0$, $\exists y \in E \ni |x-y| < \delta$.

43) Fact: Every $x \in E$ is a point of closure of E.

44) Def. The set of points of closure of E is $\overline{E}$ = *closure of* E.

45) Fact: $x \in \overline{E}$ iff $\forall \delta > 0$, $(x - \delta, x + \delta) \cap E \neq \emptyset$ iff $\exists x_n \in E \ni x_n \to x$ where $x_n \in E$ is a sequence.

46) Fact: by 43), $E \subseteq \overline{E}$.

47) Fact: $x \notin \overline{E} \Rightarrow \exists \delta > 0 \ni (x - \delta, x + \delta) \cap E = \emptyset$.

48) Def. A set F is a closed set or closed if $F = \overline{F}$.

49) By 46), F is closed iff $\overline{F} \subseteq F$.

50) i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.
ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

51) Th. $\overline{E}$ is a closed set.
(Hence $\overline{\overline{E}} = \overline{E}$.)

52) If E_1 and E_2 are closed sets, then $E_1 \cup E_2$ is a closed set.
(Hence a finite union of closed sets is a closed set.)

53) Th. The intersection of any collection $\mathcal{C}$ of closed sets is closed.
In other words, let $\mathcal{C} = \{B_\lambda, \lambda \in \Lambda\}$ be closed sets. Then $\cap_{\lambda in \Lambda} B_\lambda$ is a closed set.

54) Th. Let O be and open set and C a closed set. Then O^c is closed and C^c is open.

55) Def. A set E is called a dense set in B if $\overline{E} = B$.

56) Fact: $\mathbb{Q}$ is a dense set in $\mathbb{R}$.

57) Def. A collection $\mathcal{C} = \{E_\lambda, \lambda \in \Lambda\}$ *covers* set F if $F \subseteq \cup_{\lambda\in\Lambda} E_\lambda$.
$F \subseteq \cup_{\lambda\in\Lambda} O_\lambda$ is an *open covering* of F.
If $\mathcal{C}$ contains only a finite number of sets, then $\mathcal{C}$ is a *finite covering*.

58) **Know: Heine Borel Theorem**: Let F be a closed and bounded set of real numbers. Then each open covering of F has a finite subcovering.

In other words, if $F \subseteq \cup_{\lambda \in \Lambda} O_\lambda$, then there is a finite collection $\{O_1, ..., O_n\}$ of sets in Λ $\ni F \subseteq \cup_{i=1}^n O_i$ (where n depends on F and Λ.)

59) Lindelöf Open Covering Theorem: Let $\mathcal{C} = \{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets of real numbers. Then there is a countable subcollection $\{O_i\}$ of $\mathcal{C}$ such that $\cup_{\lambda \in \Lambda} O_\lambda = \cup_{i=1}^\infty O_i$.

60) Cor. If a set E can be covered by a collection $\mathcal{C}$ of open sets, then E can be covered by a countable collection of open sets in $\mathcal{C}$.

61) The *Cantor set* $C \subseteq [0,1]$ consists of ternary expansions $0.a_1a_2a_3...$ where $a_i \in \{0,2\}$ that do not have $a_i = 0 \; \forall i > N$ for some N. So there is a 1–1 correspondence with unique binary expansions. It can be shown that the Cantor set C is closed, uncountable, and that there is a 1–1 correspondence between C and [0,1]. Also, the ("length of C") Lebesgue measure $m(C) = 0$.

Note: a ternary expansion $0.a_1a_2a_3...$ has $a_i \in \{0,1,2\}$.

62) Def. Let $f : E \to Y \subseteq \mathbb{R}$.

a) f is *continuous at a point* $x \in E$ if $\forall \epsilon > 0$, $\exists \delta > 0 \ni \forall y \in E$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

b) f is continuous on $A \subseteq A$ if f is continuous for all $x \in A$.

c) f is continuous if f is continuous on its domain E.

(The δ depends on both x and ϵ.)

63) Def. Let $f : E \to Y \subseteq \mathbb{R}$.

Then f is *uniformly continuous on* E if $\forall \epsilon > 0$, $\exists \delta > 0 \ni \forall x, y \in E$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon$.

(The δ depends on ϵ, but is independent of $x, y \in E$.

64) Def. A sequence of functions f_n, defined on set E, *converges pointwise* on E to a function f if $\forall x \in E$, $\lim_{n \to \infty} f_n(x) \to f(x)$.

Thus $\forall x \in E$ and $\forall \epsilon > 0$, $\exists N \ni \forall n \geq N$, $|f(x) - f_n(x)| < \epsilon$.

($E \subseteq$ domain of f_n for $n = 1, 2, 3, ...$ Define the function $f : E \to \mathbb{R}$ by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in E$.)

65) Def. A sequence of functions f_n, defined on set E, *converges uniformly* on E to a function f if $\forall \epsilon > 0$, $\exists N \ni \forall n \geq N$ and $\forall x \in E$, $|f(x) - f_n(x)| < \epsilon$.

($N = N_\epsilon$ is free of x. We could define the sequence $f_n : E \to Y_n$, or have $E \subseteq$ domain of f_n for each n. Again, $f : E \to \mathbb{R}$ is defined by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in E$.)

66) Let $\mathcal{C}_O$ be the class of all open real sets. Let $\mathcal{C}_C$ be the class of all closed real sets. Let $\mathcal{C}_{CI}$ be the class of all closed intervals in $\mathbb{R} = X$. Let $\mathcal{C}_{OI}$ be the class of all open intervals in $\mathbb{R}$. Let $\mathcal{C}_{(a,b]}$ be the class of all half open intervals of the form $(a,b]$ in $\mathbb{R}$. Let $\mathcal{C}_{[a,b)}$ be the class of all half open intervals of the form $[a,b)$ in $\mathbb{R}$. Then the Borel σ-algebra $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_O)$, and it can be shown that $\mathbb{B}(\mathbb{R}) = \sigma(\mathcal{C}_C) = \sigma(\mathcal{C}_{CI}) = \sigma(\mathcal{C}_{OI}) = \sigma(\mathcal{C}_{(a,b]}) = \sigma(\mathcal{C}_{[a,b)})$.

Proof technique: If $\mathcal{C} \subseteq \sigma$-algebra $\mathcal{F}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$.

Be able to prove some of these results, such as $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_C)$ and $\sigma(\mathcal{C}_O) = \sigma(\mathcal{C}_{OI})$.

See the Math 501 qual review problems on my Math 501 webpage.