Math 501 Exam 1 is Wednesday, Feb. 19, 3:00-3:50 NO NOTES. CHECK FORMULAS: YOU ARE RESPONSIBLE FOR ANY ERRORS ON THIS HANDOUT!

1) The **universal set** X is the set of all elements under consideration. Subsets of X are of interest. The **empty set** is $\emptyset$. Let $A \subseteq X$ and $B \subseteq X$. Then the **complement of** A is $A^c = \{x \in X : x \notin A\} = \sim A = X - A$. The *difference* $B - A = \{x : x \in B$ and $x \notin A\} = B \cap A^c$. The *symmetric difference* $A \triangle B = (A \cap B^c) \cup (B \cap A^c) =$ set of all points that belong to one or the other of both sets but not to both. $[A^c]^c = A$ and $\emptyset = X^c$.

2) **Def.** Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq X$. Then $\{A_\lambda\}_{\lambda \in \Lambda}$ is an indexed family of sets.

a) The **union** $\bigcup_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for at least one $\lambda \in \Lambda\}$.

b) The **intersection** $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x \in X : x \in A_\lambda$ for all $\lambda \in \Lambda\}$.

Notation: a) Often "$\in X$" will be omitted. Hence
$\{x \in X : x \in A_\lambda$ for all $\lambda \in \Lambda\} = \{x : x \in A_\lambda$ for all $\lambda \in \Lambda\}$.

b) Often $\Lambda = \mathbb{N} = \{i\}_{i=1}^\infty = \{1, 2, ...,\} =$ the set of positive integers $=$ the set of *natural numbers*. Then $\bigcup_{\lambda \in \mathbb{N}} A_\lambda = \bigcup_{i=1}^\infty A_i$, and $\bigcap_{\lambda \in \mathbb{N}} A_\lambda = \bigcap_{i=1}^\infty A_i$.

c) If $\Lambda = \{i\}_{i=m}^\infty = \{m, (m+1), ...,\} =$ the set of integers $\geq m$, then
$\bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{i=m}^\infty A_i$, and $\bigcap_{\lambda \in \Lambda} A_\lambda = \bigcap_{i=m}^\infty A_i$.

Warning: Since ∞ is not an integer, there is no set A_∞ in $\bigcup_{i=m}^\infty A_i$ or $\bigcap_{i=m}^\infty A_i$.

3) One way to prove $A = B$ is to prove $A \subseteq B$ and $B \subseteq A$. This technique is equivalent to i) showing that if $x \in A$, then $x \in B$, and ii) showing that if $x \in B$, then $x \in A$. A second way to prove $A = B$ is to show $x \in A$ iff $x \in B$ where "iff" means "if and only if."

4) De Morgan's Laws: Let Λ be a **nonempty** index set of sets $A_\lambda \subseteq \Omega$.
i) $[\cup_{\lambda \in \Lambda} A_\lambda]^c = \cap_{\lambda \in \Lambda} A_\lambda^c$. ii) $[\cap_{\lambda \in \Lambda} A_\lambda]^c = \cup_{\lambda \in \Lambda} A_\lambda^c$. iii) $[\cap_{i=1}^\infty A_i]^c = \cup_{i=1}^\infty A_i^c$.
iv) $[\cup_{i=1}^\infty A_i]^c = \cap_{i=1}^\infty A_i^c$. v) $[A \cup B]^c = A^c \cap B^c$. vi) $[A \cap B]^c = A^c \cup B^c$.

5) Let $f : X \to Y$ be a function where X is the *domain* of f.
The *range* of f is the set $\{y \in Y : \exists x \ni y = f(x)\}$.
The function f is *onto* Y of the range of $f = Y$.
The function f is *one to one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

6) Let $f : X \to Y$, $A \subseteq X$, and $B \subseteq Y$.
i) Def. The **image** under f of A is **the set**
$f[A] = \{y \in Y : y = f(x)$ for some $x \in A\}$.
ii) Def. The **inverse image** of B is **the set**
$f^{-1}[B] = \{x \in X : f(x) \in B\}$.

Warning: $f[A]$ and $f^{-1}[B]$ are sets that depends on the function f. The inverse function need not exist.

7) Theorem: Let A_λ be sets for $\lambda \in \Lambda$.
a) $f[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f[A_\lambda]$. b) $f[\cap_{\lambda \in \Lambda} A_\lambda] \subseteq \cap_{\lambda \in \Lambda} f[A_\lambda]$. c) $f^{-1}[\cup_{\lambda \in \Lambda} A_\lambda] = \cup_{\lambda \in \Lambda} f^{-1}[A_\lambda]$.
d) $f^{-1}[\cap_{\lambda \in \Lambda} A_\lambda] = \cap_{\lambda \in \Lambda} f^{-1}[A_\lambda]$. e) $f^{-1}[A^c] = [f^{-1}[A]]^c$.

8) Def. Let $X \neq \emptyset$. A nonempty class $\mathcal{C}$ of subsets of X is an **algebra** on X (or field) if a1) $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$
a2) $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$.
An algebra is closed under complements, finite intersections, and finite unions. $X, \emptyset \in \mathcal{C}$.

9) **Know**: Def. Let $X \neq \emptyset$. A nonempty class $\mathcal{F}$ of subsets of X is a $\sigma-$*algebra* on X (or $\sigma-$field) if i) $A_1, A_2, ... \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.
ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.
Note that i) and ii) mean that a σ-algebra is an algebra. A $\sigma-$algebra is closed under countable unions, countable intersections, and complementation. The term "on X" is often understood and omitted. **Common error:** Use n instead of ∞ in i).

10) **Def.** Let $\mathcal{A}$ be a class of subsets of $X \neq \emptyset$. The $\sigma-$*algebra generated by* $\mathcal{A}$, denoted by $\sigma(\mathcal{A})$ is the intersection of all $\sigma-$algebras containing $\mathcal{A}$. Then $\sigma(\mathcal{A})$ is the smallest $\sigma-$algebra containing $\mathcal{A}$.

11) **Def.** Let $\mathcal{A}$ be the class of all open intervals (a, b) in [0,1]. Then $\sigma(\mathcal{A}) = \mathcal{B}[0,1]$ is the Borel $\sigma-$algebra on [0,1].
Fact: $\mathcal{B}[0,1] = \sigma(\mathcal{A})$ where $\mathcal{A}$ is the class of all closed intervals in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $(a, b]$ in [0,1], or $\mathcal{A}$ is the class of all intervals of the form $[a, b)$ in [0,1].

12) Def. $A_n \uparrow A$ means $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \cup_{i=1}^{\infty} A_i$.
$A_n \downarrow A$ means $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \cap_{i=1}^{\infty} A_i$. See 16).

13) Def. Let A_n be a sequence of sets.
$\overline{lim}\ A_n = limsup_n\ A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \{x : x \in A_n$ for infinitely many $A_n\}$.
$\underline{lim}\ A_n = liminf_n\ A_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \{x : x \in A_n$ for all but finitely many $A_n\}$.
If $A_n \in \mathcal{F}$, then $\overline{lim}\ A_n, \underline{lim}\ A_n \in \mathcal{F}$. Also, $liminf_n\ A_n \subseteq limsup_n\ A_n$.

14) Def. If $liminf_n\ A_n = limsup_n\ A_n$, then $lim_n\ A_n = A = liminf_n\ A_n = limsup_n\ A_n$, written $A_n \to A$. (The subscript n is sometimes omitted.)

15) **Theorem.** Let A_n be a sequence of $\mathcal{F}$ sets.
a) $\overline{lim}\ A_n, \underline{lim}\ A_n \in \mathcal{F}$.
b) If $lim_n\ A_n$ exists, then $lim_n\ A_n = A \in \mathcal{F}$.
c) $liminf_n\ A_n \subseteq limsup_n\ A_n$.
d) $(limsup_n\ A_n)^c = liminf_n\ A_n^c$.
e) $(liminf_n\ A_n)^c = limsup_n\ A_n^c$.

Proof. a) $C_n = \cup_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n. Hence $\cap_{n=1}^{\infty} C_n = \overline{lim}\ A_n \in \mathcal{F}$. $B_n = \cap_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n. Hence $\cup_{n=1}^{\infty} B_n = \underline{lim}\ A_n \in \mathcal{F}$.

b) Follows from a).

c) If $x \in A_n$ for all but finitely many A_n, then $x \in A_n$ for all but infinitely many A_n. Hence if $x \in liminf_n\ A_n$ then $\omega \in limsup_n\ A_n$. Thus $liminf_n\ A_n \subseteq limsup_n\ A_n$.

d) By De Morgan's laws applied twice, $(limsup_n\ A_n)^c = [\cap_{n=1}^{\infty} C_n]^c = \cup_{n=1}^{\infty} C_n^c = liminf_n A_n^c$ where C_n is given in a).

e) By De Morgan's laws applied twice, $(liminf_n\ A_n)^c = [\cup_{n=1}^{\infty} B_n]^c = \cap_{n=1}^{\infty} B_n^c = limsup_n A_n^c$ where B_n is given in a). $\square$

Remark 1.6. a) If $limsup_n\ A_n \subseteq A \subseteq liminf_n\ A_n$, then $lim_n A_n = A$ by Theorem in point 15).
b) $B_n = \cap_{k=n}^{\infty} A_k \uparrow \underline{lim}\ A_n$. Thus $lim_{n\to\infty} \cap_{k=n}^{\infty} A_k = \underline{lim}\ A_n$.
c) $C_n = \cup_{k=n}^{\infty} A_k \downarrow \overline{lim} A_n$. Thus $lim_{n\to\infty} \cup_{k=n}^{\infty} A_k = \overline{lim}\ A_n$, and $\overline{lim}\ A_n = \cap_{n=1}^{\infty} C_n$.
d) **Do not treat convergence of sets like convergence of functions**. $A_n \to A$ iff $limsup_n\ A_n = liminf_n\ A_n$ which implies that if $\omega \in A_n$ for infinitely many n, then $\omega \in A_n$ for all but finitely many n.

e) **Warning:** Students who have not figured out the examples 16) and 17) tend to make errors on similar problems.
f) Typically want to show that open, closed, and half open intervals can be written as a countable union or countable intersection of intervals of another type. Then the Borel σ-field $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$ where $\mathcal{C}$ is a class of intervals such as the class of all open intervals.

16) **Example.** Prove the following results.
a) $A_1 \subseteq A_2 \subseteq \cdots$ implies that $A_n \uparrow A = \cup_{n=1}^{\infty} A_n$.
b) $A_1 \supseteq A_2 \supseteq \cdots$ implies that $A_n \downarrow A = \cap_{n=1}^{\infty} A_n$.

Proof. a) For each n, $A = \cup_{k=n}^{\infty} A_k$. Thus $limsup\ A_n = \cap_{n=1}^{\infty} A = A$. For each n, $\cap_{k=n}^{\infty} A_k = A_n$. Thus $liminf\ A_n = \cup_{n=1}^{\infty} A_n = A$.

b) For each n, $\cup_{k=n}^{\infty} A_k = A_n$. Thus $limsup\ A_n = \cap_{n=1}^{\infty} A_n = A$. For each n, $\cap_{k=n}^{\infty} A_k = A$. Thus $liminf\ A_n = \cup_{n=1}^{\infty} A = A$.

17) **Example.** Simplify the following sets where $a < b$. Answers might be $(a,b), [a,b)$, $(a,b], [a,b], [a,a] = \{a\}, (a,a) = \emptyset$.

a) $I = \displaystyle\bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$, b) $I = \displaystyle\bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right]$, c) $I = \displaystyle\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$,

d) $I = \displaystyle\bigcap_{n=1}^{\infty} \left[a, b + \frac{1}{n}\right)$, e) $I = \displaystyle\bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n}\right)$, f) $I = \displaystyle\bigcup_{n=1}^{\infty} \left[a, b - \frac{1}{n}\right]$.

Solution. a) $I = (a,b] = \displaystyle\bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right) = \bigcap_{n=1}^{\infty} A_n$ where $A_n \downarrow I$. Note that $(a,b] \subseteq A = \displaystyle\bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$ since $b \in \left(a, b + \dfrac{1}{n}\right)$ $\forall n$. For any $\epsilon > 0$, $(a, b+\epsilon] \not\subseteq A$ since $b + 1/n < b + \epsilon$ for large enough n. Note that $b + 1/n \to b$, but sets are not functions. (A common error is to say $I = (a,b)$.)

b) $I = (a,b) = \displaystyle\bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} A_n$ where $A_n \uparrow I$. Note that $b \notin \displaystyle\bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] = A$ since $b \notin \left(a, b - \dfrac{1}{n}\right]$ $\forall n$ and since $n \in \mathbb{N}$ so $n = \infty$ never occurs. Note that $\left(a, b - \dfrac{1}{n}\right] = \emptyset$ if $b - 1/n \leq a$. For any $\epsilon > 0$ such that $b - \epsilon > a$, it follows that $(a, b-\epsilon] \in A$ since $b - 1/n > b - \epsilon$ for large enough n, say $n > N_\epsilon$. Thus $b - \epsilon \in A$ all but finitely many times.

c) $I = (a,b) = \displaystyle\bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} A_n$ where $A_n \uparrow I$. Note that $a, b \notin A = I$ since $a, b \notin \left[a + \dfrac{1}{n}, b - \dfrac{1}{n}\right]$ $\forall n \in \mathbb{N}$. Then the proof is similar to that of b).

d) $I = [a,b] = \displaystyle\bigcap_{n=1}^{\infty} \left[a, b + \frac{1}{n}\right) = \bigcap_{n=1}^{\infty} A_n$ where $A_n \downarrow I$. This proof is similar to that of a).

e) $I = [a,a] = \{a\} = \displaystyle\bigcap_{n=1}^{\infty} \left[a, a + \frac{1}{n}\right) = \bigcap_{n=1}^{\infty} A_n$ where $A_n \downarrow I$. Note that $a \in A = I$, but $a + \epsilon \notin A$ $\forall \epsilon > 0$.

f) $I = [a,b) = \bigcup_{n=1}^{\infty} \left[a, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} A_n$ where $A_n \uparrow I$. This proof is similar to that of b).

Chapter 2:

18) Let $S \subseteq \mathbb{R}$. Then d is an upper bound for S if for each $x \in S$, $x \leq d$. Then $c = sup\ S = sup_{x \in S}\ x$ is the least upper bound for S if $sup\ S$ is an upper bound for S, and if $sup\ S \leq d$ for any upper bound for S. Similarly, a is a lower bound for S if for each $x \in S$, $x \geq a$. Then $b = inf\ S = inf_{x \in S}\ x$ is the greatest lower bound for S if $inf\ S$ is a lower bound for S, and if $inf\ S \geq a$ for any lower bound for S. Both $sup\ S$ and $inf\ S$ are unique when they exist.

Facts: I) $sup\ S = c \in \mathbb{R}$ iff i) $\forall x \in S, x \leq c$, ii) $\forall \epsilon > 0, \exists x_0 \in S \ni c - \epsilon < x_0 \leq c$.
II) $inf\ S = b \in \mathbb{R}$ iff i) $\forall x \in S, x \geq b$, ii) $\forall \epsilon > 0, \exists x_0 \in S \ni b \leq x_0 < b + \epsilon$.

19) **Completeness Axiom**: Every nonempty set S of real numbers which has an upper bound has a least upper bound.

20) $inf_{x \in S}\ x = -sup_{x \in S}\ -x$. Thus any nonempty set S of real numbers which has a lower bound has a greatest lower bound.

21) Let $\{x_n\}_{n=1}^{\infty} = \{x_n\}$ be a sequence of real numbers.
a) $x_n \uparrow x$ means $x_1 \leq x_2 \leq \cdots$ and $x_n \to x$. (nondecreasing sequence)
b) $x_n \downarrow x$ means $x_1 \geq x_2 \geq \cdots$ and $x_n \to x$. (nonincreasing sequence)
(These monotone sequences have "limits" if $\pm\infty$ are allowed. So $x_n \uparrow \infty$ means the sequence diverges to ∞, while $x_n \downarrow \infty$ means the sequence diverges to $-\infty$. If x is a real number, then x is the limit of the sequence.)

22) The limit superior and limit inferior of a sequence will be useful. The sequence $\{a_n\}_{n=1}^{\infty} = a_1, a_2, ...$ Let $\{a_n\}_{n=m}^{\infty} (= a_m, a_{m+1}, ...)$ be a sequence of numbers. Then
i) sup a_n = least upper bound of $\{a_n\}$, and
ii) inf a_n = greatest lower bound of $\{a_n\}$.

23) **Def.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

a) The limit superior of the sequence $limsup_n a_n = \overline{lim}_n a_n = lim_n\ sup_{k \geq n}\ a_k$ is the limit of the nonincreasing sequence $\{\sup_{k \geq j} a_k\}_{j=1}^{\infty}$.

b) The limit inferior of the sequence $liminf_n a_n = \underline{lim}_n a_n = lim_n\ inf_{k \geq n} a_k$ is the limit of the nondecreasing sequence $\{\inf_{k \geq j} a_k\}_{j=1}^{\infty}$.

24) **Remark.** a) Unlike the limit, $\overline{lim}_n a_n$ and $\underline{lim}_n a_n$ always exist when $\pm\infty$ are allowed as limits, since limits of nondecreasing and nonincreasing sequences then exist.

b) $\underline{lim}_n a_n \leq \overline{lim}_n a_n$

c) $lim_n a_n = a$ iff $\underline{lim}_n a_n = \overline{lim}_n a_n = a$. Hence the limit of a sequence exists iff $\underline{lim}_n a_n = \overline{lim}_n a_n$. Again, $a = \pm\infty$ is allowed.

d) Let $lim_n^* a_n$ be $\overline{lim}_n a_n$ or $\underline{lim}_n a_n$.
If $a_n \leq b_n$, then $lim_n^* a_n \leq lim_n^* b_n$.
If $a_n < b_n$, then $lim_n^* a_n \leq lim_n^* b_n$.
If $a_n \geq b_n$, then $lim_n^* a_n \geq lim_n^* b_n$.
If $a_n > b_n$, then $lim_n^* a_n \geq lim_n^* b_n$.
That is, when taking the $liminf$ or $limsup$ on both sides of a strict inequality, the $<$ or $>$ must be replaced by $\leq$ or $\geq$.

A similar result holds for limits if both limits exist.

e) $limsup_n(-a_n) = -liminf_n a_n$.
f) i) $limsup_n a_n = \overline{lim}_n a_n$ is the limit of the nonincreasing sequence

$$\sup_{k \geq m} a_k, \quad \sup_{k \geq m+1} a_k, \ldots.$$

ii) $liminf_n a_n = \underline{lim}_n a_n$ is the limit of the nondecreasing sequence

$$\inf_{k \geq m} a_k, \quad \inf_{k \geq m+1} a_k, \ldots.$$

iii)

$$\overline{lim}_n a_n = \inf_n \sup_{k \geq n} a_k = \lim_{k \to \infty} \sup(a_n, n \geq k).$$

iv)

$$\underline{lim}_n a_n = \sup_n \inf_{k \geq n} a_k = \lim_{k \to \infty} \inf(a_n, n \geq k).$$

v) If a limit point of a sequence $\{a_n\}$ is any number, including $\pm\infty$, that is a limit of some subsequence, then $liminf_n a_n$ and $limsup_n a_n$ are the inf and sup of the set of limit points, often the smallest and largest limit points.

25) $x_0 \in \mathbb{R}$ is a cluster point of $\{x_n\}$ if $\forall \epsilon > 0$ and $\forall N \in \mathbb{N}, \exists n \geq N \ni |x_n - x_0| < \epsilon$. Thus infinitely many terms of the sequence $\{x_n\}$ are within ϵ of x_0.
∞ is a cluster point of $\{x_n\}$ if given $\beta > 0$ and given N, $\exists n \geq N \ni x_n > \beta$. Thus infinitely many terms of the sequence $\{x_n\}$ are $> \beta$.
$-\infty$ is a cluster point of $\{x_n\}$ if given $\alpha < 0$ and given N, $\exists n \geq N \ni x_n < \alpha$. Thus infinitely many terms of the sequence $\{x_n\}$ are $< \alpha$.
A limit point is also called an accumulation point and a cluster point. If $\{x_n\}$ is a bounded sequence, then $\overline{lim}\ x_n =$ largest accumulation point (cluster point) of $\{x_n\}$, and $\underline{lim}\ x_n =$ smallest accumulation point of $\{x_n\}$.
$x_0 \in \mathbb{R}$ is a cluster point of $\{x_n\}$

26) **Warning:** A common error is to take the limit of both sides of an equation $a_n = b_n$ or of an inequality, e.g. $a_n \leq b_n$. Taking the limit is an error if the existence of the limit has not been shown. If $\pm\infty$ are allowed, $\underline{lim}_n a_n$ and $\overline{lim}_n a_n$ always exits. Hence the $\underline{lim}_n a_n$ or $\overline{lim}_n a_n$ of the above equation or inequality can be taken.