M501 Exam 2 is Friday, March. 28, 3:00-3:50 NO NOTES. CHECK FORMULAS: YOU ARE RESPONSIBLE FOR ANY ERRORS ON THIS HANDOUT!

ch. 3

67) Notation: Unless stated otherwise, measure and measurable mean Lebesgue measure and Lebesgue measurable for chapters 2-10. We will usually use $m(A)$ for the Lebesgue measure of set A and $\mu(A)$ for a general measure of A.

68) Let $\mathbb{P}(X)$ be the largest $\sigma-$algebra of all subsets of the universal set X.

69) For the measure and outer measure with $X = \mathbb{R}$, let $A \subseteq \mathbb{R}$ and $I \subseteq \mathbb{R}$ be understood.

70) In the following definition 71), let $\mathcal{C}(A)$ consist of all countable collections of open intervals that contain (cover) A.
For an interval such as the open interval $I = (a, b)$ or [a,b] or [a,b) or [a,b], let the length of the interval $l(I) = b - a$ where $a \leq b$ with $b = \infty$ and $a = -\infty$ possible.

71) Def. Let A be a set and let $\mathcal{C}(A) = \{\{I_n\}_{n=1}^N, N \leq \infty, A \subseteq \cup_{n=1}^N I_n,$ where I_n is an open interval $\forall n \in \mathbb{N}\}$. Then the (Lebesgue) **outer measure** of a set A is $m^*(A) = \inf_{\mathcal{C}(A)} \sum_{n=1}^\infty l(I_n)$.

72) The set function $m^* : \mathbb{P}(\mathbb{R}) \to [0, \infty]$ where $[0, \infty] = [0, \infty) \cup \{\infty\}$.

73) **Know**: Properties of m^*.
i) $m^*(\emptyset) = 0$.
ii) *monotonicity*: Let $A \subseteq B$, then $m^*(A) \leq m^*(B)$.
iii) the outer measure of an interval I is the length of the interval: $m^*(I) = l(I)$.
iv) *countable subadditivity*: Let $A_1, A_2, ...$ be a sequence of sets. Then
$m^*(\cup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty m^*(A_n)$.
v) *finite subadditivity*: $m^*(\cup_{i=1}^N A_i) \leq \sum_{i=1}^N m^*(A_i)$.
vi) countable sets have outer measure equal to 0: If set E is countable, then $m^*(E) = 0$.

Be able to prove v) and vi), and to use vi) to show that nonempty intervals are uncountable sets.

74) Def. A set $G \in G_\delta$ if G is a countable intersection of open sets.

75) Prop 3.5. a) Given any set A and any $\epsilon > 0$, there is an open set O such that $A \subseteq O$ and $m^*(O) \leq m^*(A) + \epsilon$.
b) There is $G \in G_\delta$ such that $m^*(G) = m^*(A)$.

76) A set function $\mu : \mathcal{F} \to [0, \mu(X)]$ is *countably additive* if $\mu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$ if $E_1, E_2, ...$ are **disjoint sets** in the $\sigma-$algebra $\mathcal{F}$, and $\mu(X) = \infty$ is allowed where X is the universal set.

77) The outer measure m^* is not countably additive on $\mathbb{P}(\mathbb{R}) = \sigma-$algebra of all subsets of $\mathbb{R}$.

78) **Know**: Def. A set E is (Lebesgue) *measurable* if for any set A, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$.

79) **Know**: Let $\mathcal{F}_M$ be the class of measurable sets. Thus E is measurable iff $E \in \mathcal{F}_M$.
Properties:
i) E is measurable iff E^c is measurable.
ii) $\emptyset$ and $\mathbb{R} = X$ are measurable.
iii) If $E_1, E_2, ...$ are measurable, then $\cup_{i=1}^\infty E_i$ is measurable.
iv) By i)–iii), $\mathcal{F}_M$ is a $\sigma-$algebra.

v) If $m^*(E) = 0$, then E is measurable.
vi) Every open set O is measurable.
vii) $\mathbb{B}(\mathbb{R}) \subseteq \mathcal{F}_M$. Hence every Borel set is measurable, including closed sets.

80) **Know**: If E is a measurable set, the (Lebesgue) **measure** of E is $m(E) = m^*(E)$. Thus m is m^* restricted to the $\sigma-$algebra $\mathcal{F}_M$ or to the Borel $\sigma-$algebra $\mathbb{B}(\mathbb{R}) \subseteq \mathcal{F}_M$. Thus the measure of an interval is the length of the interval.

81) **Know**: Properties of m.
i) *monotonicity*: If $A \subseteq B$ are measurable, then $m(A) \leq m(B)$.
ii) *countable subadditivity*: Let $E_1, E_2, ...$ be measurable. Then $m(\cup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty m(E_n)$.
iii) *finite subadditivity*: Let $E_1, E_2, ..., E_N$ be measurable. Then $m(\cup_{i=1}^N E_i) \leq \sum_{i=1}^N m(E_i)$.
iv) *countable additivity*: Let $E_1, E_2, ...$ be disjoint measurable sets. Then $m(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty m(E_n)$.
v) *finite additivity*: Let $E_1, E_2, ..., E_N$ be disjoint measurable sets. Then $m(\cup_{n=1}^N E_n) = \sum_{n=1}^N m(E_n)$.
vi) If $m^*(E) = 0$, then $m(E) = 0$. Hence $m(E) = 0$ if E is countable. If E contains a nonempty interval, then E is not countable. Note that the Cantor set C is uncountable with $m(C) = 0$.

82) If $m(E) = m^*(E) = 0$, then E is known as a *null set* or a *set of measure* 0. Null sets include the empty set, countable sets, and the uncountable Cantor set.

83) **Know**: It can be shown that there exist sets U, U_i that are not measurable and that m^* is not countably additive on $\mathbb{P}(\mathbb{R})$. Note that $U_i \notin \mathbb{B}(\mathbb{R}) \subseteq \mathcal{F}_M$.
$V =$ the Vitali set is a nonmeasurable set.

84) The Lebesgue measure on sets such as $X = [0, 1]$ and $X = [0, \infty)$ is similar. Note that if $X = [0, 1]$, then measurable sets A satisfy $m(A) \leq m([0, 1]) = 1$.

85) **Know**: Def. An extended real valued function $f : D \to \mathbb{R}^*$ is a (Lebesgue) **measurable function** if its domain D is measurable and if for each $a \in \mathbb{R}$, the set $f^{-1}[(a, \infty)] = \{x \in D : f(x) > a\}$ is a measurable set.

Note: From exam 1 review point 7), for $f : X \to Y$ with X the universal set, i) $f^{-1}[\cup_i A_i] = \cup_i f^{-1}[A_i]$, ii) $f^{-1}[\cap_i A_i] = \cap_i f^{-1}[A_i]$, and iii) $f^{-1}[A^c] = (f^{-1}[A])^c$.

Note: Real functions are extended real value functions. We could say that $f(x)$ is a measurable function on D.

86) Th. An extended real valued function $f : D \to \mathbb{R}^*$ is a (Lebesgue) **measurable function** if its domain D is measurable and if any one of the following three conditions hold. a) For each $a \in \mathbb{R}$, the set $f^{-1}[[a, \infty)] = \{x \in D : f(x) \geq a\}$ is a measurable set.
b) For each $a \in \mathbb{R}$, the set $f^{-1}[(-\infty, a)] = \{x \in D : f(x) < a\}$ is a measurable set.
c) For each $a \in \mathbb{R}$, the set $f^{-1}[(-\infty, a])] = \{x \in D : f(x) \leq a\}$ is a measurable set.

Note: For proving that $f(x)$ is a measurable function on D, it is useful to act as if D is the universal set. Then the complement of a set in 86) or 85) is the complement with respect to D.

87) Almost any function you can think of, before taking a measure theory course, is a measurable function.

88) Let the characteristic function = indicator function = $\chi_A(x) = I_A(x) = 1$ for $x \in A$ and $I_A(x) = 0$ for $x \notin A$.

89) **Know**: Th. $I_A(x)$ is a measurable function iff A is measurable.
($I_A : \mathbb{R} \rightarrow \{0,1\}$.)

90) Thus a nonmeasurable function is $I_V(x)$ where V is a nonmeasurable set such as the Vitali set.

91) Let $\mathcal{L}(D) = \{$ all real valued L. measurable functions with domain D $\}$.
Let $\overline{\mathcal{L}}(D) = \{$ all extended real valued L. measurable functions with domain D $\}$.
Note that D is a measurable set by the definition of a measurable function.

92) Let $h = f \cdot g$ be the product function $h(x) = f(x)g(x)$.

93) Th. Let $c \in \mathbb{R}, f \in \mathcal{L}(D)$, $g \in \mathcal{L}(D)$, and $f_i \in \mathcal{L}(D)$. Then
i) $c + f \in \mathcal{L}(D)$.
ii) $f + g \in \mathcal{L}(D)$. Hence $\sum_{i=1}^n f_i \in \mathcal{L}(D)$.
iii) $f - g \in \mathcal{L}(D)$.
iv) $f \cdot g \in \mathcal{L}(D)$. (product function of 92)) Hence $f_1 \cdots f_n \in \mathcal{L}(D)$.
v) If $g(x) \neq 0 \ \ \forall x \in D$, then $1/g \in \mathcal{L}(D)$.
vi) $cf \in \mathcal{L}(D)$.

Thus $h = cI_{\mathbb{R}} \in \mathcal{L}(D)$ where $h(x) = c \ \ \forall x$ is a constant function.

94) For the note under 85), $f : X \rightarrow Y$ with X the universal set. Now suppose $f : D \rightarrow Y$ where the domain D is not necessarily equal to X. Then
$\{x \in D : f(x) \in A^c\} = D - \{x \in D : f(x) \in A\} = D \cap \{x \in D : f(x) \in A\}^c$.

95) Th. Let $c \in \mathbb{R}, f \in \mathcal{L}(D)$, $g \in \mathcal{L}(D)$, and $f_i \in \mathcal{L}(D)$.
a) $\max(f,g) \in \mathcal{L}(D)$. Hence $\max(f_1, ..., f_n) \in \mathcal{L}(D)$.
b) $\min(f,g) \in \mathcal{L}(D)$. Hence $\min(f_1, ..., f_n) \in \mathcal{L}(D)$.
c) $sup_n \ f_n \in \mathcal{L}(D)$.
d) $inf_n \ f_n \in \mathcal{L}(D)$.
e) $limsup_n \ f_n \in \mathcal{L}(D)$.
f) $liminf_n \ f_n \in \mathcal{L}(D)$.
g) If $lim_n f_n = f$, then $f \in \mathcal{L}(D)$.
h) $|f| \in \mathcal{L}(D)$.
i) If h is continuous on D, then $h \in \mathcal{L}(D)$.

96) Th. If $f_1(u)$ is continuous and $u = f_2(x)$ is measurable, then the composite function $f_1 \circ f_2 = f_1(f_2(x))$ is measurable.

Thus a continuous function of a measurable function is measurable. However, a measurable function of a measurable function need not be measurable.

97) **Know**: A property is said to hold **almost everywhere** (ae or a.e.) if the set of points where the property fails to hold is a set of measure 0.

For example, if $lim_n f_n = f$ ae, then $f \in \mathcal{L}(D)$.

98) **Know**: Th. Let $f \in \mathcal{L}(D)$ and let $f(x) = g(x)$ ae. Then $g \in \mathcal{L}(D)$.

99) Littlewood's 3 principles ($X = \mathbb{R}$): a) Every measurable set in nearly a finite union of intervals.
b) Every measurable function is nearly continuous.
c) Every convergent sequence of measurable functions is nearly uniformly convergent.

100) **Know: Egoroff's Theorem** If $\{f_n\}$ is a sequence of measurable functions that converge to a real valued function f ae on a measurable set E, then given $\eta > 0$, $\exists A \subseteq E$ with $m(A) < \eta$ such that f_n converges uniformly to f on $E - A$.

101) Def. The function ϕ is a **simple function** if $\phi(x) = \sum_{i=1}^n \alpha_i \chi_{A_i}(X)$ where the $A_i \subseteq \mathbb{R}$, $A_i \in \mathcal{F}_M$, and $\alpha_i \in \mathbb{R}$.

102) Thus a simple function is a measurable function. So the function in 90) is not a simple function.

ch. 4

103) Sometimes we will use $\int_a^b f(x)dx$ as the Lebesgue integral and $(R) \int_a^b f(x)dx$ as the Riemann integral, but sometimes the (R) will be omitted.

104) Riemann integral: Let f be a bounded function on $[a, b]$. Let partition $\Delta = \{t_0, t_1, ..., t_n\}$ with $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$. Let $M_i = \sup_{x \in [t_{i-1}, t_i]} f(x)$.
Let $m_i = \inf_{x \in [t_{i-1}, t_i]} f(x)$.
Let the *upper sum* $= U(\Delta) = U(f, \Delta) = \sum_{i=1}^n M_i(t_i - t_{i-1})$.
Let the *lower sum* $= L(\Delta) = L(f, \Delta) = \sum_{i=1}^n m_i(t_i - t_{i-1})$.
Let the *upper integral* $= \overline{\int_a^b} f(x)dx = \inf_\Delta U(\Delta)$.
Let the *lower integral* $= \underline{\int_a^b} f(x)dx = \sup_\Delta L(\Delta)$.

105) Def. Let f be a bounded function on $[a, b]$. Then f is *Riemann integrable* iff $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$.

106) Let $E_i = [t_{i-1}, t_i]$. Define $\int \chi_{E_i}(x)dx = m(E_i) = t_i - t_{i-1}$. Then the step function $\phi(x) = \sum_{i=1}^n M_i \chi_{E_i}(x) \geq f(x)$ and the step function $\psi(x) = \sum_{i=1}^n m_i \chi_{E_i}(x) \leq f(x)$.

107) Def. $\int_a^b \phi(x)dx = \sum_{i=1}^n M_i \int \chi_{E_i}(x)dx = \sum_{i=1}^n M_i(t_i - t_{i-1}) = U(\Delta)$.

108) Th. The bounded function f is Riemann integrable on $[a, b]$ iff
$\inf_{\text{step}\phi \geq \text{f}} \int_a^b \phi(x)dx = \sup_{\text{step}\psi \leq \text{f}} \int_a^b \psi(x)dx$.

109) As in 101), a function $\phi(x)$ is a simple function if ϕ is measurable and ϕ only assumes finitely many values. Let the set of nonzero values of ϕ be $\{a_1, .., a_n\}$. Let $A_i = \{x \in \mathbb{R} : \phi(x) = a_i\}$. Then the A_i are disjoint, measurable sets ($\in \mathcal{F}_M$).

110) Def. The *canonical representation* of ϕ is $\phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$.

111) The canonical representation of ϕ is unique, but can be hard to find.

112) Def. Let ϕ be a simple function vanishing outside a set of finite measure. Let $\phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ be the canonical representation of ϕ. Then (the L. integral)
$\int \phi(x)dx = \int \phi = \sum_{i=1}^n a_i \; m(A_i)$.

113) Suppose that E is a measurable set so that χ_E is a simple function. Then $\int \chi_E = \int \chi_E(x)dx = m(E)$.

Note: The following theorem removes the restriction of using the canonical representation since we could have $a_i = a_j$ for $i \neq j$. The assumptions in 114) make ϕ a simple function.

114) Th. Let $\phi = \sum_{i=1}^n a_i \; \chi_{E_i}$ with $a_i \neq 0$ and $E_i \cap E_j = \emptyset$ for $i \neq j$. Suppose $E_i \in \mathcal{F}_M$ and $m(E_i) < \infty$. Then
$\int \phi = \sum_{i=1}^n a_i \; m(E_i)$.

115) Th. Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then $\int(a\phi + b\psi) = a \int \phi + b \int \psi$.

Note: Then by induction, if ϕ_i are simple functions which vanish outside a set of finite measure, then $\int \sum_{i=1}^n a_i \phi_i = \sum_{i=1}^n a_i \int \phi_i$.

Note: In 116) the E_i need not be disjoint, and 116) is *linearity for step functions.*

116) Let $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ vanish outside a set of finite measure. Then $\int \phi = \sum_{i=1}^n a_i m(E_i)$.

117) Def. For any $E \in \mathcal{F}_m$, $\int_E \phi = \int \phi \; \chi_E$ where ϕ is a simple function that vanishes outside a set of finite measure.

118) (monotonicity for simple functions): Let ϕ and ψ be simple functions which vanish outside a set E of finite measure. If $\phi \geq \psi$ ae, then $\int_E \phi \geq \int_E \psi$.

119) If $A, B \in \mathcal{F}_M$, $A \cap B = \emptyset$, and ϕ is defined as in 116), then $\int_{A \cup B} \phi = \int_A \phi + \int_B \phi$.

120) Prop 4.3. Th. Let f be defined and bounded on a measurable set E with $m(E) < \infty$. Then $f \in \mathcal{L}(E)$ iff $\inf_{simple \;\, \psi \geq f} \int_E \psi(x)dx = \sup_{simple \;\, \phi \leq f} \int_E \phi(x)dx$.

121) **Know**: Def. Let f be a bounded measurable function on E with $m(E) < \infty$. Then the *L. integral* $\int_E f(x)dx = \inf_{simple \;\, \psi \geq f} \int_E \psi(x)dx = \sup_{simple \;\, \phi \leq f} \int_E \phi(x)dx$.

Note: i) For 121), 120) forces $f \in \mathcal{L}(E)$ by 120).
ii) If $E = [a,b]$, then $\int_E f(x)dx = \int_E f = \int_a^b f(x)dx = \int_a^b f$.
iii) $\int_E f = \int f \chi_E = \int_E f(x)dx$.
iv) To show equality in 121), it is enough to show
$\inf_{simple \;\, \psi \geq f} \int_E \psi(x)dx \leq \sup_{simple \;\, \phi \leq f} \int_E \phi(x)dx$ since
$\inf_{simple \;\, \psi \geq f} \int_E \psi(x)dx \geq \sup_{simple \;\, \phi \leq f} \int_E \phi(x)dx$.

122) (**L. integral for bounded measurable functions on** E): Suppose f_i, f and g are bounded measurable functions on E with $m(E) < \infty$.
i) (linearity): $\int_E (af + bg) = a \int_E f + b \int_E g$.
Hence $\int_E \sum_{i=1}^n a_i f_i = \sum_{i=1}^n a_i \int_E f_i$.
ii) If $f = g$ ae on E, then $\int_E f = \int_E g$.
iii) (monotonicity): If $f \leq g$ ae on E, then $\int_E f \leq \int_E g$.
In particular, $|\int_E f| \leq \int_E |f|$.
iv) If $a \leq f(x) \leq b$ on E, then $a \; m(E) \leq \int_E f \leq b \; m(E)$.
v) If $A, B \in \mathcal{F}_M$, $A \cap B = \emptyset$ and $A \cup B = E$, then $\int_{A \cup B} f = \int_A f + \int_B f$.

123) Th. Let f be a bounded function on $[a,b]$. If f is Riemann integrable on $[a,b]$, then f is L. integrable on $[a,b]$.

(Thus f is L. measurable on $[a,b]$.)

124) Prop 4.7 Th. A bounded function f on $[a,b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

(Thus f is continuous ae on $[a,b]$.)