M501 Exam 3 is Friday, April 25, 3:00-3:50 NO NOTES. CHECK FORMULAS: YOU ARE RESPONSIBLE FOR ANY ERRORS ON THIS HANDOUT!

From Exam 2 review, know ch. 4 material 103)-124)

ch. 4

125) **Know: Bounded Convergence Theorem (BCT):** Let $E \in \mathcal{F}_M$, $m(E) < \infty$, and $f_n \in \mathcal{L}(E)$. If $\exists M > 0$ such that $|f_n(x)| \leq M$ $\forall n \in \mathbb{N}$ and $\forall x \in E$, and if $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$, then $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$.

Remark: Now we go from bounded measurable functions on E where $m(E) < \infty$ to nonnegative measurable functions on E where $m(E) = \infty$ is possible.

126) **Know:** Def. Let $E \in \mathcal{F}_M$, $f, h \in \mathcal{L}(E)$, $f \geq 0$ in E, and h bounded in E. Then the *L. integral* $\int_E f = sup_{h \leq f} \int_E h$ where $m(\{x \in E : h(x) \neq 0\}) < \infty$.

127) Let $B_E(h) = B_E$ denote that $h \in \mathcal{L}(E)$, h is bounded in E, and $m(\{x \in E : h(x) \neq 0\}) < \infty$. Then for measurable set E and measurable function $f \geq 0$ in E, denote the integral by $\int_E f = sup_{h \leq f, B_E} \int_E h$.

128) **Know**. Def. A nonnegative measurable function f is (L.) *integrable* over $E \in \mathcal{F}_M$ if $\int_E f < \infty$.

129) **Properties of the L. integral for nonnegative functions**: Let $E \in \mathcal{F}_M$, $f, f_i, g \in \mathcal{L}(E)$, $f, f_i, g \geq 0$ in E.
i) $\int_E f \geq 0$.
ii) If $m(E) = 0$, then $\int_E f = 0$.
iii) For $c > 0$, $\int_E cf = c \int_E f$.
iv) (restricted linearity for nonnegative functions):
$\int_E (f + g) = \int_E f + \int_E g$.
If $a_i > 0$, then $\int_E \sum_{i=1}^n a_i f_i = \sum_{i=1}^n a_i \int_E f_i$.
v) (monotonicity): If $f \leq g$ ae on E, then $\int_E f \leq \int_E g$.
Also, if g is integrable, then f is integrable, and $\int_E (g - f) = \int_E g - \int_E f$.

130) **Know: Fatou's Lemma**: If f_n be a sequence of nonnegative measurable functions in E and if $f_n(x) \to f(x)$ ae in E, then $\displaystyle\int_E f \leq \underline{lim} \int_E f_n$.

131) **Know: Monotone Convergence Theorem (MCT)**: Let $0 \leq f_n \leq f_{n+1}$ be a sequence of nonnegative measurable functions in $E \in \mathcal{F}_M$. If $f_n(x) \to f(x)$ ae in E, then $\displaystyle\int_E f_n \to \int_E f$.

Remark: In the MCT, the f_n form an increasing sequence (hence a monotone sequence) of nonnegative functions: $0 \leq f_n \uparrow f$ ae in E, and $\lim_{n\to\infty} \int_E f_n = \int_E f = \int_E \lim_{n\to\infty} f_n$.

132) **Know**: Th. Let $u_i \geq 0$ be measurable functions over $E \in \mathcal{F}_M$. Let $f(x) = \sum_{i=1}^\infty u_i(x)$ for $x \in E$. Then $\int_E f =$

$$\int_E \sum_{i=1}^\infty u_i(x)dx = \sum_{i=1}^\infty \int_E u_i(x)dx.$$

133) Th. Let $f \geq 0$ be measurable over E. Let $E_i \in \mathcal{F}_M$ be disjoint with $E = \cup_{i=1}^\infty E_i$. Then

$$\int_{\cup_{i=1}^\infty E_i} f = \sum_{i=1}^\infty \int_{E_i} f.$$

134) Th. Let $f \geq 0$ and integrable over E in $\mathcal{F}_M$. Then for any $\epsilon > 0$, $\exists \delta > 0$ such that for any set $A \subseteq E$ with $m(A) < \delta$, $\int_A f \leq \epsilon$.

General L. Integral

135) **Know**: Def. If $f : D \to [-\infty, \infty]$, then the **positive part** $f^+ = fI(f \geq 0) = max(f, 0)$, and the **negative part** $f^- = -fI(f \leq 0) = max(-f, 0) = -min(f, 0)$. Hence $f^+(x) = f(x)I(f(x) \geq 0)$ and $f^-(x) = -f(x)I(f(x) \leq 0)$ for $x \in D$.

Here $I(f \geq 0) = I(0 \leq f \leq \infty)$ while $I(f(x) \leq 0) = I(-\infty \leq f \leq 0)$. If f is measurable, then $f^+ \geq 0$, $f^- \geq 0$ are both measurable, $f = f^+ - f^-$, and $|f| = f^+ + f^-$.

136) **Know**: Def. A measurable function f is L. integrable over a measurable set E if f^+ and f^- are both L. integrable over E.

Remark. Thus $\int_E f^+ < \infty$ and $\int_E f^- < \infty$.

137) **Know**: Th. A measurable function f is L. integrable over a measurable set E iff $|\int_E f| < \infty$ iff $\int_E |f| < \infty$.

Remark: Hence f is integrable on E iff f is *absolutely integrable* on E.

138) **Know**: Def. If the measurable function f is L. integrable over $E \in \mathcal{F}_M$, then the L. integral is $\displaystyle\int_E f = \int_E f^+ - \int_E f^-$.

Remark: So the L. integral $\int_E f$ exists iff $|\int_E f| < \infty$.

139) Unless otherwise stated, assume $f(x)$, $g(x)$, and $f_i(x)$ are measurable and Lebesgue integrable, and that all indicated sets are measurable. Note that a property holding everywhere, eg $f(x) \leq g(x)$ for all $x \in E$, is a special case of the property holding almost everywhere.

a) $\int_E cf(x)dx = c\int_E f(x)dx$ for any constant c.

b) $\int_E cdx = c\ m(E)$ for any constant c. (True even if $m(E) = \infty$ so χ_E is not L. integrable.)

c) If $m(E) = 0$, then $\int_E f(x)dx = 0$.

d) *mean-value theorem for Lebesgue integrals*: If $a \leq f(x) \leq b$ for all $x \in E$, then $a\ m(E) \leq \int_E f(x)dx \leq b\ m(E)$.

e) If $E = E_1 \cup E_2$ where $E_1 \cap E_2 = \emptyset$, then $\int_E f(x)dx = \int_{E_1} f(x)dx + \int_{E_2} f(x)dx$.

f) If $E = \cup_{i=1}^n E_i$ where the E_i are disjoint, then $\int_E f(x)dx = \sum_{i=1}^n \int_{E_i} f(x)dx$.

g) $\int_E [f(x) + g(x)]dx = \int_E f(x)dx + \int_E g(x)dx$.

h) (**linearity**): $\int_E \sum_{i=1}^n f_i(x)dx = \sum_{i=1}^n \int_E f_i(x)dx$.

i) (**monotonicity**): If $f(x) \leq g(x)$ almost everywhere on E, then $\int_E f(x)dx \leq \int_E g(x)dx$.

j) A measurable function f is Lebesgue integrable on E iff $|f|$ is Leb. integrable on E.

k) If $f(x)$ is Lebesgue integrable on E, then $|\int_E f(x)dx| \leq \int_E |f(x)|dx$.

l) If $f(x) = g(x)$ almost everywhere on E, then $\int_E f(x)dx = \int_E g(x)dx$.

m) If $f(x) \geq 0$ ae on E and $\int_E f(x)dx = 0$, then $f(x) = 0$ ae on E.

n) If $f(x)$ and $g(x)$ are bounded and measurable on E, then $h(x) = f(x)g(x)$ is Lebesgue integrable on E with $\int_E f(x)g(x)dx < \infty$.

o) If $|f(x)| \leq g(x)$ ae on E where g is integrable on E, then f is integrable on E and $\int_E |f| \leq \int_E g$.

p) If $\int_E f$ exists, then $f(x)$ is finite ae on E.

q) If $\int_E f$ exists, and $A \subseteq E$, then $\int_A f$ exists and $\int_A |f| \leq \int_E |f|$.

r) If f is integrable and g is bounded on E, then $h(x) = f(x)g(x)$ is integrable on E.

140) **Know**: Lebesgue Convergence Theorem = Lebesgue's Dominated Convergence Theorem (LDCT): Let g be integrable over $E \in \mathcal{F}_M$, and let f_n be a sequence of measurable functions over E such that $|f_n(x)| \le g(x)$ on E and $f(x) = \lim_{n\to\infty} f_n(x)$ ae on E. Then $\displaystyle\int_E f = \int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_E f_n$.

ch. 5

141) Def. Let function $f : [a,b] \to \mathbb{R}$ and let $\pi = a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be a partition of $[a,b]$. Let $r^+ = r$ if $r \ge 0$ and $r^+ = 0$ if $r < 0$. Let $r^- = |r| - r^+$.
Let $p = \ ^+V_a^b(f,\pi) = \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^+$.
Let $n = \ ^-V_a^b(f,\pi) = \sum_{i=1}^k [f(x_i) - f(x_{i-1})]^-$.
Let $t = V_a^b(f,\pi) = \sum_{i=1}^k |f(x_i) - f(x_{i-1})|$.
Let $P = P_a^b = \ ^+V_a^b(f) = \sup\{\ ^+V_a^b(f,\pi), \pi$ a partition of $[a,b]\}$ = positive variation of f over $[a,b]$.
Let $N = N_a^b = \ ^-V_a^b(f) = \sup\{\ ^-V_a^b(f,\pi), \pi$ a partition of $[a,b]\}$ = negative variation of f over $[a,b]$.
Let $T = T_a^b = \ V_a^b(f) = \sup\{\ V_a^b(f,\pi), \pi$ a partition of $[a,b]\}$ = total variation of f over $[a,b]$.
Then f is of *bounded variation* over $[a,b]$, written f is BV, if $T = \ V_a^b(f) < \infty$.

Remarks: a) p, n, t, P, N and T are all nonnegative. b) Consider $t = t(f,\pi) = \sum_{i=1}^k |f(x_i) - f(x_{i-1})|$. If the sum is bounded for all possible partitions π, then f is BV.

141) Lemma 5.4. Th. Suppose function $f : [a,b] \to \mathbb{R}$ is of bounded variation over $[a,b]$. Then a) $T = V_a^b(f) =^+ V_a^b(f) +^- V_a^b(f) = P + N$.
b) $f(b) - f(a) =^+ V_a^b(f) -^- V_a^b(f) = P - N$.

142) Def. Let $f : [a,b] \to \mathbb{R}$. Then a) f is *nondecreasing* if $f(x) \le f(y)$ for $x \le y$, b) f is *nonincreasing* if $f(x) \ge f(y)$ for $x \le y$, c) if f is nondecreasing or nonincreasing, the f is a *monotone function*, d) f is *increasing* if $f(x) < f(y)$ for $x < y$, e) f is *decreasing* if $f(x) > f(y)$ for $x < y$.

Remark: Let f be differentiable on open interval I. If $f'(x) > 0$ on I, then f is an increasing function on I. If $f'(x) < 0$ on I, then f is an decreasing function on I. If $f(x) = c$ on I where c is a constant, then f is both a nondecreasing and a nonincreasing function on I. A similar result holds for $[a,b]$ if $I = (a,b)$ and f is continuous on $[a,b]$.

143) Def. A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* on $[a,b]$ if given $\epsilon > 0$, $\exists \delta > 0$ such that $\sum_{i=1}^n |f(x_i') - f(x_i)| < \epsilon$ for every finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with $\sum_{i=1}^n |x_i' - x_i| < \delta$.

144) Remarks: i) If f is absolutely continuous on $[a,b]$, then f is continuous and of bounded variation on $[a,b]$. Also f has a derivative f' ae on $[a,b]$.
ii) If f and g are absolutely coninuous, then so are $f - g$, $f + g$, the product function $f(x)g(x)$ and $f(x)/g(x)$ provided that $g(x) \ne 0$ on $[a,b]$.

145) Def. If f ia a L. integrable function on $[a,b]$, then its *indefinite integral* is $F(x) = \int_a^x f(t)dt$ for $x \in [a,b]$.

146) Th. A function F is an indefinite integral iff F is absolutely continuous.

Remark: Hence F is continuous and of bounded variation by 144).

147) Th. Suppose $F(x) = \int_a^x f(t)dt + F(a)$ for $x \in [a,b]$. If either a) f is bounded and measurable on $[a,b]$ or b) f is integrable on $[a,b]$, then $F'(x) = f(x)$ ae in $[a,b]$.

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148) Def. The space of all functions $f(x)$ for which $|f(x)|^p$, $p \geq 1$, is L. integrable on $[a,b]$, i.e. $\int_a^b |f(x)|^p dx < \infty$, is the L^p space: $L^p = L^p[a,b]$.

149) a) The $L^1[a,b]$ space consists of all L. integrable functions on $[a,b]$ since $\int_a^b |f| < \infty$.

b) The $L^2[a,b]$ space is the Hilbert space with functions f such that $\int_a^b [f(x)]^2 dx < \infty$. Such functions are square integrable.

c) If A is a measurable set, $L^p(A)$ consists of the L. integrable functions f such that $\int_A |f(x)|^p dx < \infty$,

150) Let $\|f\|_p = [\int_a^b |f|^p]^{1/p}$.

151) Cauchy Schwarz Inequality:

$$\left|\int_a^b f(x)g(x)dx\right| \leq \sqrt{\int_a^b |f(x)|^2 dx}\sqrt{\int_a^b |g(x)|^2 dx}$$

where $f, g \in L^2$.

152) Holder's Inequality:

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $f \in L^p$, and $g \in L^q$.

153) Minkowski's Inequality:

$$\int_a^b |f(x) + g(x)|^p dx \leq \left(\int_a^b |f(x)|^p dx\right)^{1/p} + \left(\int_a^b |g(x)|^p dx\right)^{1/p}$$

where $p \geq 1$, $f, g \in L^p$.

154) Th. If $1 \leq p \leq q$, then $L^q \subseteq L^p \subseteq L^1$.

Remark. So if $f \in L^q$, then $f \in L^p$ and $f \in L^1$.

155) Th. If $f \in L^1$, $g \in L^p$ and $|f| \leq |g|$ for x in $[a,b]$, then $f \in L^p$.

156) Def. Let $f_n, f \in L^p[a,b]$. If $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p dx = 0$, then f_n *converges in mean* to f in the space L^p.

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Remark: Recall that X is the universal set.

157) Def. a) A *measurable space* is $(X, \mathcal{F})$ consisting of the set X and a $\sigma-$algebra $\mathcal{F}$.

b) A set $A \subseteq X$ is *measurable* (wrt to $\mathcal{F}$) if $A \in \mathcal{F}$.

Remark: Often $\mathcal{F} = \mathcal{B}(X)$, the Borel $\sigma-$algebra on X. The set A is also called a measurable set.

158) **Know:** Def. A *measure* μ on a measurable space $(X, \mathcal{F})$ is

i) a nonnegative set function defined for all subsets of $\mathcal{F}$.

ii) $\mu(\emptyset) = 0$

iii) (**countable additivity**): Let $E_1, E_2, ...$ be disjoint measurable sets. Then $\mu(\cup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$.

159) More properties of a measure $\mu : \mathcal{F} \to [0, \mu(X)]$ where $0 < \mu(X) \leq \infty$ depends on μ. iv) *monotonicity*: If $A \subseteq B$ are measurable, then $\mu(A) \leq \mu(B)$.
v) *countable subadditivity*: Let $E_1, E_2, ...$ be measurable. Then
$\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.
vi) *finite subadditivity*: Let $E_1, E_2, ..., E_N$ be measurable. Then $\mu(\cup_{i=1}^{N} E_i) \leq \sum_{i=1}^{N} \mu(E_i)$.
vii) *finite additivity*: Let $E_1, E_2, ..., E_N$ be disjoint measurable sets. Then
$\mu(\cup_{i=1}^{N} E_i) = \sum_{i=1}^{N} \mu(E_i)$.
viii) If the $E_i \in \mathcal{F}$ (so the E_i are measurable), $\mu(E_1) < \infty$, and $E_{i+1} \subseteq E_i$, then $\mu(\cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_n)$.

160) **Know**: Def. An extended real valued function $f : D \to \mathbb{R}^*$ is a **measurable function** if its domain D is measurable and if for each $a \in \mathbb{R}$, the set $f^{-1}[(a, \infty)] = \{x \in D : f(x) > a\}$ is a measurable set ($\in \mathcal{F}$).

Note: Real functions are extended real value functions. We could say that $f(x)$ is a measurable function on D.

161) Th. An extended real valued function $f : D \to \mathbb{R}^*$ is a **measurable function** if its domain D is measurable and if any one of the following three conditions hold. a) For each $a \in \mathbb{R}$, the set $f^{-1}[[a, \infty)] = \{x \in D : f(x) \geq a\}$ is a measurable set ($\in \mathcal{F}$).
b) For each $a \in \mathbb{R}$, the set $f^{-1}[(-\infty, a)] = \{x \in D : f(x) < a\}$ is a measurable set ($\in \mathcal{F}$).
c) For each $a \in \mathbb{R}$, the set $f^{-1}[(-\infty, a])] = \{x \in D : f(x) \leq a\}$ is a measurable set ($\in \mathcal{F}$).

162) μ is a complete measure if $B \in \mathcal{F}$, $A \subseteq B$, and $\mu(B) = 0$ implies $A \in \mathcal{F}$.

163) Assume μ is a complete measure unless stated otherwise. Assume measurable sets and mmeasurable functions are with respect to (wrt) $(X, \mathcal{F}, \mu)$. Notation: $\int_E f = \int_E f \; d\mu = \int_E f(x) \; d\mu(x)$.

164) **Know: Fatou's Lemma**: If f_n be a sequence of nonnegative measurable functions in E and if $f_n(x) \to f(x)$ ae in E, then $\displaystyle\int_E f \; d\mu \leq \underline{lim} \int_E f_n \; d\mu$.

165) **Know: Monotone Convergence Theorem (MCT)**: Let $0 \leq f_n$ be a sequence of nonnegative measurable functions in $E \in \mathcal{F}$. If $f_n(x) \to f(x)$ ae in E and if $f_n \leq f$ in E for all n, then $\displaystyle\int_E f_n \; d\mu \to \int_E f \; d\mu$.

Remark. Do not need $f_n \uparrow f$ in the MCT.

165) **Properties of the integral for nonnegative functions**: Let $E \in \mathcal{F}$, f, f_i, g nonnegative measurable functions over E. i) $\int_E f \; d\mu \geq 0$.
ii) (restricted linearity for nonnegative functions):
$\int_E (f + g) \; d\mu = \int_E f \; d\mu + \int_E g \; d\mu$.
If $a_i > 0$, then $\int_E \sum_{i=1}^{n} a_i f_i \; d\mu = \sum_{i=1}^{n} a_i \int_E f_i \; d\mu$.
iii) A nonnegative, measurable function f is integrable over $E \in \mathcal{F}$ if $\int_E f \; d\mu < \infty$.

166) Def. A measurable function f is integrable over a measurable set E if f^+ and f^- are both integrable over E.

Remark. Thus $\int_E f^+ \; d\mu < \infty$ and $\int_E f^- \; d\mu < \infty$.

167) **Know**: Th. A measurable function f is integrable over a measurable set E iff $|\int_E f \; d\mu| < \infty$ iff $\int_E |f| \; d\mu < \infty$.

Remark: Hence f is integrable on E iff f is *absolutely integrable* on E.

168) **Know**: Def. If the measurable function f is integrable over $E \in \mathcal{F}$, then the

integral is $\displaystyle\int_E f\ d\mu = \int_E f^+\ d\mu - \int_E f^-\ d\mu$.

Remark: So the integral $\int_E f$ exists iff $|\int_E f| < \infty$.

169) **Properties of the integral**: Assume $(X, \mathcal{F}, \mu)$ is a complete measure space (so μ is a complete measure). Unless otherwise stated, assume f, g, f_i are integrable over $E \in \mathcal{F}$. Note that a property holding everywhere, eg $f(x) \leq g(x)$ for all $x \in E$, is a special case of the property holding almost everywhere.

a) $\int_E cf(x)\ d\mu = c \int_E f(x)\ d\mu$ for any constant c.
b) $\int_E c\ d\mu = c\ \mu(E)$ for any constant c. (True even if $\mu(E) = \infty$ so χ_E is not integrable.)
c) $\int_E [af(x) + bg(x)]\ d\mu = a \int_E f(x)\ d\mu + b \int_E g(x)\ d\mu$.
d) (**linearity**): $\int_E \sum_{i=1}^n a_i f_i(x)\ d\mu = \sum_{i=1}^n a_i \int_E f_i(x)\ d\mu$.
e) (**monotonicity**): If $f(x) \leq g(x)$ almost everywhere on E, then $\int_E f(x)\ d\mu \leq \int_E g(x)\ d\mu$.
f) If $|h| \leq |f|$ on E and h is measurable, then h is integrable on E.
g) If $f(x)$ is integrable on E, then $|\int_E d(x)\ d\mu| \leq \int_E |f(x)|\ d\mu$.

170) **Know**: Lebesgue Convergence Theorem = Lebesgue's Dominated Convergence Theorem (LDCT): Let g be integrable over $E \in \mathcal{F}$, and let f_n be a sequence of measurable functions over E such that $|f_n(x)| \leq g(x)$ on E and $f(x) = \lim_{n\to\infty} f_n(x)$ ae on E. Then $\displaystyle\int_E f\ d\mu = \int_E \lim_{n\to\infty} f_n\ d\mu = \lim_{n\to\infty} \int_E f_n\ d\mu$.

Not on Exam 3 but on the final and Quiz 11:

$\oint$ **11.6**:

171) Def. Let $(X, \mathcal{F}, \mu)$ and $(X, \mathcal{F}, \nu)$ be two measure spaces with the same X and $\mathcal{F}$. i) Then μ and ν are *mutually singular* if there are disjoint sets $A, B \in \mathcal{F}$ such that $X = A \cup B$ and $\nu(A) = \mu(B) = 0$.
ii) A measure ν is *absolutely continuous* wrt μ, written $\nu << \mu$, if $\nu(A) = 0$ for any set $A \in \mathcal{F}$ for which $\mu(A) = 0$.

172) Th. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $f \geq 0$ be a measurable function on X. For each $E \in \mathcal{F}$, define $\nu(E) = \int_E f \; d\mu$. Then ν is a measure on $(X, \mathcal{F})$.

173) (Let $(X, \mathcal{F}, \mu)$ be a measure space.) Def. A measure μ is *σ-finite* if there is a sequence of measurable sets $X_i \in \mathcal{F}$ such that $X = \cup_{i=1}^{\infty} X_i$ and each $\mu(X_i) < \infty$.

174) a) If $\mu(X) = c$ for some positive real number c, then μ is a *finite measure* and hence a σ-finite measure.
b) The sequence X_i in 173) can be taken to be disjoint.

175) **Radon-Nikodym Theorem**: Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{F}$ which is absolutely continous wrt μ. Then there is a nonnegative function f such that for each $E \in \mathcal{F}$, $\nu(E) = \int_E f \; d\mu$. The function f is "unique" in that if g is any measurable function with this property, then $f = g$ ae μ.

176) The function f in 175) is called the Radon-Nikodym derivative of ν wrt μ and is sometimes denoted by $f = \dfrac{d\nu}{d\mu}$.

$\oint$ **12.4**:

177) Def. Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two measure spaces. Let the Cartesian product = direct product = cross product $X \times Y = \{(x, y) : x \in X, y \in Y\}$. If $A \subseteq X$ and $B \subseteq Y$, then $A \times B$ is called a *rectangle.* If $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$, then $A \times B$ is called a *measurable rectangle.* If $A \times B$ is a measurable rectangle, then the product measure $\lambda = \mu \times \nu$ satisfies $\lambda(A \times B) = \mu(A)\nu(B)$.

178) Def. Let $\mathcal{A}$ be the collection of all measurable rectangles: $\mathcal{A} = \{A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$. Let $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 = \sigma(\mathcal{A})$ be the smallest σ-algebra containing $\mathcal{A}$.

179) Th. $(X \times Y, \mathcal{F}, \lambda)$ is a measure space.

Remark: In the following theorem, "almost all" means ae.

180) **Fubini's Theorem**: Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two complete measure spaces and f an integrable function on $X \times Y$. Then
i) for almost all x, the function $f_x(y) = f(x, y)$ is an integrable function on Y.
ii) For almost all y, the function $f^y(x) = f(x, y)$ is an integrable function on X.
iii) $\int_Y f(x, y) \; d\nu(y)$ is an integrable function on X.
iv) $\int_X f(x, y) \; d\mu(x)$ is an integrable function on Y.
v) $\int_X [\int_Y f \; d\nu] \; d\mu = \int_Y [\int_X f \; d\mu] \; d\nu = \int_{X \times Y} f \; d(\mu \times \nu)$.

180) **Tonelli's Theorem**: Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two σ-finite measure spaces and $f \geq 0$ a measurable function on $X \times Y$. Then
i) for almost all x, the function $f_x(y) = f(x, y)$ is an integrable function on Y.
ii) For almost all y, the function $f^y(x) = f(x, y)$ is an integrable function on X.
iii) $\int_Y f(x, y) \; d\nu(y)$ is an integrable function on X.
iv) $\int_X f(x, y) \; d\mu(x)$ is an integrable function on Y.
v) $\int_X [\int_Y f \; d\nu] \; d\mu = \int_Y [\int_X f \; d\mu] \; d\nu = \int_{X \times Y} f \; d(\mu \times \nu)$.

$\oint$ **3.6**:

181) **Lusin's Theorem**: Let f be a measurable real valued function on [a,b]. Given $\delta > 0$ there is a continuous function ϕ on [a,b] such that $m(\{x : f(x) \neq \phi(x)\}) \leq \delta$.

$\oint$ **5.1**:

182) Def. Let Λ be a collection of intervals. Then Λ covers E *in the sense of Vitali*, if for each $\epsilon > 0$ and any $x \in \Lambda$, there exists $I \in \Lambda$ such that $x \in I$ and $l(I) < \epsilon$.

Remark: i) Could let $I = I_x$. ii) Λ is also called a *Vitali covering* of E.

183) **Vitali's Covering Lemma**: Let E be a set of finite outer measure and Λ a collection of intervals that covers E in the sense of Vitali. Then given $\epsilon > 0$, there exists a finite disjoint collection $\{I_1, ..., I_N\}$ of intervals in Λ such that
$m^*(E - \cup_{i=1}^N I_i) < \epsilon$.

$\oint$ **4.5**:

184) Def. **convergence ae**: Let the f_n and f be measurable functions. Let $f_n : D \to \mathbb{R}$, $f : D \to \mathbb{R}$, and $f_n \to f$ ae in D. Then $m(\{x \in D : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$.

185) Remark: In 184) and 186), m can be replaced by μ. If f_n converges to f in measure, then $f_n \to f$ in measure-μ.

186) Def. Let the f_n and f be measurable functions. Let $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$. and $f_n \to f$ ae in D. Then f_n *converges in measure* to f if $\forall\epsilon > 0$,
$\lim_{n\to\infty} m(\{x \in D : \lim_{n\to\infty} |f_n(x) - f(x)| \geq \epsilon\}) = 0$.

187) Given measurable space $(X, \mathcal{F}, \mu)$ where the f_n and f are measurable functions, if $f_n \to f$ ae in D, then $f_n \to f$ in measure-μ.

188) Convergence in measure does not imply convergence ae.

189) Usual convergence (convergence everywhere) and uniform convergence are special cases of convergence ae. Hence if f_n does converge to f in measure, then f_n does not converge to f uniformly.

190) Given measurable space $(X, \mathcal{F}, \mu)$, if $f \in L^1$ and $f_n \in L^p$, then f_n *converges to* f in L^p if $\int_X |f_n - f|^p \, d\mu \to 0$ as $n \to \infty$.

191) If $f_n \to f$ in L^p, then $f_n \to f$ in measure.

192) Chebyshev's inequality: Let f be a nonnegative measurable function on $(X, \mathcal{F}, \mu)$. If $0 < p < \infty$ and $0 < \epsilon < \infty$, then

$$\mu(\{x \in X : f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \int_X [f(x)]^p \, d\mu.$$