1] Theory of Calculus
Intro to Analysis
M352 |x| → Intro to Analysis
||x|| M452

↓

Real Analysis
Lebesgue Measure and Integration
M501

↓

Functional Analysis
M502
huge

Probability and Measure
M581

2] Want to know when limits and integrals can be interchanged:

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int \underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)}\,dx = \int f(x)\,dx.$$

The regularity conditions are much simpler for Lebesgue integration than for Riemann integration.
(Leh beg)

3] A measure μ is a set function.
mu

For an interval $A=(a,b)$, Lebesgue measure
$\lambda(A) = b-a$ = length of interval.

§ 1.1 4] often denote a $\underbrace{\text{universal set}}_{}$ set of all elements of interest
↑section by $X = \mathbb{X}$, $Y = \mathbb{Y}$, $Z = \mathbb{Z}$, capital letters.

5] Subsets of $\mathbb{X}$ are often denoted by

A, B, C etc.

6] Exceptions: $\mathbb{R} = R = (-\infty, \infty)$ = set of real numbers = {all real numbers}

, $\phi = \{ \}$ = empty set = set with no elements. $\mathbb{Z}$ = set of integers

$\mathbb{N} = N = \{1, 2, \ldots\}$ = set of positive integers
boldface = set of natural numbers
(some authors include 0, so the set of nonnegative integers)

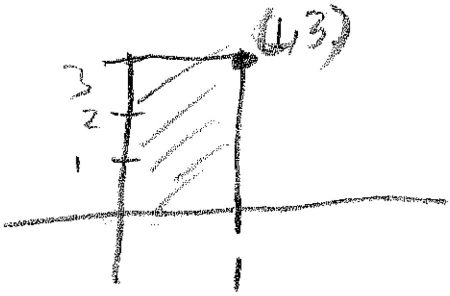
$Q = \mathbb{Q}$ = set of rational numbers

7] If $x \in A$ iff $x \in B$, then $A = B$.
if and only if

8] If $x \in A \Rightarrow x \in B$ then
A is a subset of B $\underbrace{A \subseteq B}_{\text{text}}$ or $A \subseteq B$.

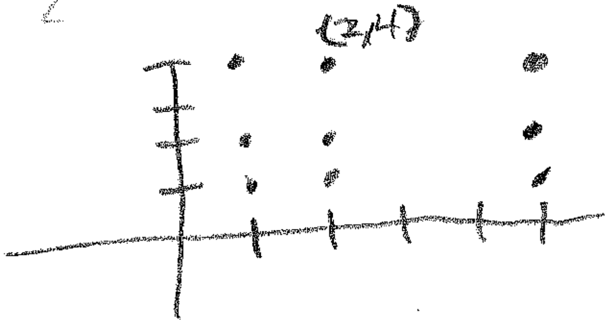
9]* p 8 If X and Y are sets, the <u>cartesian product</u> or direct product of X and Y is $X \times Y = \{(x, y)\}$ = set of all ordered pairs such that $x \in X$ and $y \in Y$.

ex) $A = [0,1]$, $B = [0,3]$



$A \times B$ = rectangle

ex) $A = \{1, 2, 5\}$ $B = \{1, 2, 4\}$



$A \times B$ = grid

10) $X \times Y \times Z = \{ (x, y, z) \}$ = set of ordered triplets such that $x \in X$, $y \in Y$, $z \in Z$

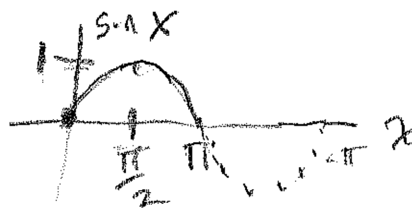
11) $A_1 \times A_2 \times \cdots \times A_k = \{ \underbrace{(a_1, a_2, \ldots, a_k)}_{} \}$. k-tuple

12) Denote the set $\{$ all elements of X which have property $P \}$ by

$\{ x \in X : P(x) \}$

ex) $\{ x \in [0, \pi] : \sin(x) < 1 \} = [0, \pi] \setminus \{ \frac{\pi}{2} \}$

$= [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$



ex} $\{x \in [0,1] : x \in \mathbb{Q}\}$ = set of all rationals between 0 and 1.

$\phi 13$

13} $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

$A \setminus B = \{x \in A : x \notin B\} = A - B$

such that $= A \cap (\sim B) = A \cap B^c$

$\mathbb{R} \setminus B = B^c = \{x \in \mathbb{R} : x \notin B\}$ if $X = \mathbb{R}$

$A \cup B = \{x \in A$ or $x \in B\}$

$x \in A$, $x \in B$ or both

$A \cap B = \{x \in A$ and $x \in B\}$

$A \times B = \{(x, y) : x \in A, y \in B\}$

ex} $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ = x-y plane

14} $A \cup \phi = A$, $A \cap \phi = \phi$

$A \cup X = X$, $A \cap X = A$

* The complement of A (relative to X) is

$\{x \in X : x \notin A\} = \sim A = A^c = X - A$

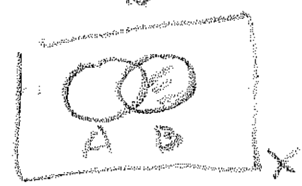
15} The difference $B-A$ (= relative complement of A in B)

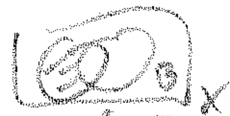
$= \{x : x \in B \text{ and } x \notin A\}$

$= B \cap A^c$

$A-B = A \cap B^c$

B−A



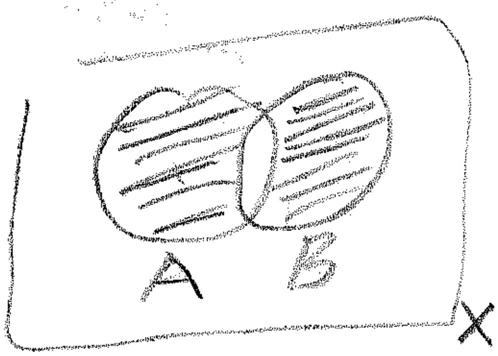


A−B

16} The *symmetric difference*

$A \,\Delta\, B = (A-B) \cup (B-A)$

$= (A \cap B^c) \cup (B \cap A^c)$ = set of all points that belong to one or the other of both sets but not to both.



$A \Delta B$ is shaded

17}* If $A \cap B = \phi$, then A and B are *disjoint*.

18) Laws:

i) $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$.

ii) $A \cup B = B \cup A$

iii) $(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C$

iv) $A \cap B = B \cap A$

v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

distributive laws

19) De Morgan's Laws

$$[A \cup B]^c = A^c \cap B^c$$

$$[A \cap B]^c = A^c \cup B^c$$

nonempty unless told otherwise

20) Let Λ be an index set for a nonempty collection of sets.

The union $\bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda} A_\lambda =$

$\{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$.

The intersection $\bigcap_{\lambda \in \Lambda} A_\lambda = \bigcap_{\lambda} A_\lambda =$

$\{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}$.

(could say $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ is the collection,

or $\bigcap_{A_\lambda \in \mathcal{C}} A_\lambda = \bigcap_{\lambda \in \Lambda} A_\lambda$)

21) * De Morgan's Laws

$$\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right]^c = \bigcap_{\lambda\in\Lambda} \left[A_\lambda\right]^c$$

De Morgan Laws: flip ∪ or ∩

$$\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right]^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$$

22) Distributive Laws

$$B \cap \left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] = \bigcup_{\lambda\in\Lambda} (B \cap A_\lambda)$$

$$B \cup \left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] = \bigcap_{\lambda\in\Lambda} (B \cup A_\lambda)$$

23) If $\Lambda = N$ use $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

24) $\bigcap_{i=1}^{n} B_i = B_1 \cap B_2 \cap \cdots \cap B_n$

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$

§1.2

25} A function f of X into Y is denoted by $f: X \to Y$ where X is the domain of f.

The range of f is the set

$$\{ y \in Y : \underset{\text{there exists}}{\exists} x \underset{\text{such that}}{\ni} y = f(x) \}.$$

26} f is onto Y if

range of $f = Y$.

27} $f: X \to Y$ is one to one if

$$f(x_1) = f(x_2) \underset{\text{implies}}{\Rightarrow} x_1 = x_2.$$

28} If $f: X \to Y$ is one to one and onto, then there exists an inverse function $f^{-1}: Y \to X$ such that $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

29} To find the inverse function $x = f^{-1}(y)$, solve the equation $y = f(x)$ for x.

ex} $y = f(x) = \ln(x)$

so $e^{y} = e^{\ln x} = x = f^{-1}(y) = e^{y}$.

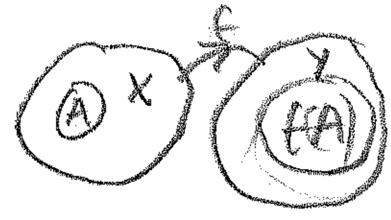
30} * p.9 Def. Let $f: X \to Y$ and $A \subseteq X$.

The <u>image</u> under f of A is <u>the set</u>

$f[A] = \{y \in Y : y = f(x) \text{ for some } x \in A\}$.

The <u>range</u> of f $= f[X]$, and

f is <u>onto</u> Y iff $Y = f[X]$.



31} know Def. Let $f: X \to Y$ and $B \subseteq Y$.

The <u>inverse image</u> of B is <u>the set</u>

$f^{-1}[B] = \{x \in X : f(x) \in B\}$.

32) Warning! $f[A]$ and $f^{-1}[A]$ are sets. The inverse function f^{-1} need not exist.

ex) $f: A \to B$

A = domain of f

$f[A] = \{ f(x) \in B : x \in A \}$ = range of A under $\underbrace{\text{map}}_{\text{function}}$ f

$f^{-1}[B] = \{ x \in A : f(x) \in B \}$.

For any $y \in B$, $\{y\} \subseteq B$ and

$f^{-1}[\{y\}] = \{ x \in A : f(x) = y \}$

If $f[A] = B$, then f is <u>onto</u>.

33) * One way to prove $A = B$ is to show

i) if $x \in A$ then $x \in B$ so $A \subseteq B$ and

ii) if $x \in B$ then $x \in A$ so $B \subseteq A$.

34] ✓ Another way to prove $A = B$ is to show $x \in A$ iff $x \in B$.

35] ✓ Theorem! Let A_λ be sets for $\lambda \in \Lambda$, nonempty

a) $f\left[\bigcup_{\lambda \in \Lambda} A_\lambda\right] = \bigcup_{\lambda \in \Lambda} f[A_\lambda]$

b) $f\left[\bigcap_{\lambda \in \Lambda} A_\lambda\right] \subseteq \bigcap_{\lambda \in \Lambda} f[A_\lambda]$ get equality iff f is 1:1

c) $f^{-1}\left[\bigcup_{\lambda \in \Lambda} A_\lambda\right] = \bigcup_{\lambda \in \Lambda} f^{-1}[A_\lambda]$

d) $f^{-1}\left[\bigcap_{\lambda \in \Lambda} A_\lambda\right] = \bigcap_{\lambda \in \Lambda} f^{-1}[A_\lambda]$

e) $f^{-1}[A^c] = \left[f^{-1}[A]\right]^c$

Proof of d) i) Let $x \in f^{-1}\left[\bigcap_\lambda A_\lambda\right]$.

Then $\exists\, y \in \bigcap_\lambda A_\lambda \ni f(x) = y$. $\Leftrightarrow$

$\forall \lambda \in \Lambda$, $\exists\, y \in A_\lambda$ with $f(x) = y$.

Thus $\forall \lambda \in \Lambda$, $x \in f^{-1}[A_\lambda]$.

Thus $x \in \bigcap_\lambda f^{-1}[A_\lambda]$. $\left(\text{So } f^{-1}\left[\bigcap_\lambda A_\lambda\right] \subseteq \bigcap_\lambda f^{-1}[A_\lambda]\right)$

ii) Let $x \in \bigcap_\lambda f^{-1}[A_\lambda] \Rightarrow$

$\forall \lambda \in \Lambda,\ x \in f^{-1}[A_\lambda] \Rightarrow$

$\forall \lambda \in \Lambda, \exists\ y_\lambda \in A_\lambda \ni f(x) = y_\lambda$.

{Since f is a function, $f(x) = y_\lambda = y\ \forall \lambda \in \Lambda$.

key trick

Thus $\forall \lambda \in \Lambda$, $y \in A_\lambda$ with $f(x) = y$.

$\therefore\ x \in f^{-1}[\bigcap_\lambda A_\lambda]$.

(So $\bigcap_\lambda f^{-1}[A_\lambda] \subseteq f^{-1}[\bigcap_\lambda A_\lambda]$.)

Then i) and ii) prove d). $\square$

36) Let $f: X \to Y$, $g: Y \to Z$.

Let $h: X \to Z$ be defined by

$h(x) = g \circ f(x) = g(f(x))$

= composition of g with f.

§1.4 37) Want collections of sets A

such that $\mu(A)$ and $\int_A f(x)\,dx$ can be computed.

An algebra does not contain enough sets, some σ-algebras do. All subsets of X has too many sets.

MS01 7

38] Def. Let $X \neq \emptyset$. A nonempty collection $\mathcal{C}$ of subsets of X is an algebra (on X) if

a1) $A \cup B \in \mathcal{C}$ whenever $A, B \in \mathcal{C}$

a2) $\overline{A} \in \mathcal{C}$ whenever $A \in \mathcal{C}$.

$\overline{A} = X - A$

39] An algebra is a nonempty collection of sets $\mathcal{C}$ that is closed under finite unions, finite intersections, and complementation.

$A_1, \ldots, A_n \in \mathcal{C} \Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{C}$, $\bigcap_{i=1}^{n} A_i \in \mathcal{C}$

and $A_i^c \in \mathcal{C}$. Also $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$.

$\exists A \in \mathcal{C} \therefore A^c \in \mathcal{C} \therefore A \cup A^c = X \in \mathcal{C} \therefore X^c = \emptyset \in \mathcal{C}$

proof sketch. Note that $A_1, \ldots, A_n \in$ algebra $\mathcal{C} \Rightarrow$

$A_1 \cup A_2 \in \mathcal{C}$, $\therefore (A_1 \cup A_2) \cup A_3 \in \mathcal{C}$,

$\ldots \therefore \bigcup_{i=1}^{n} A_i \in \mathcal{C}$. More formally, use induction.

Also $A_1^c, \ldots, A_n^c \in \mathcal{C}$. Hence $\bigcup_{i=1}^{n} A_i^c \in \mathcal{C}$.

Thus $\left[\bigcup_{i=1}^{n} A_i^c\right]^c = \bigcap_{i=1}^{n} A_i \in \mathcal{C}$.

40} Know for final | Def. Let $X \neq \emptyset$. A nonempty collection $\mathcal{F}$ of subsets of X is a σ-algebra (σ-field, sigma-algebra) (on X) if

$\sigma 1$) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$\sigma 2$) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

warning! A common error is to use n instead of ∞ in $\sigma 1$).

41} Theorem: Principle of Mathematical Induction:

Let $P(n)$ be a statement for each $n \in N$ such that a) $P(1)$ is true and

b) for each $k \in N$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in N$.

42}* There are several equivalent definitions for a σ-algebra.

A) Let $X \neq \emptyset$. A class $\mathcal{F}$ of subsets of X is a σ-algebra if i) $X \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

B) A <u>nonempty</u> class $\mathcal{F}$ of subsets of $X \neq \phi$ is a σ-algebra on X if $\mathcal{F}$ is closed under countable unions, countable intersections and complements.

proof of B) Let $A_1, A_2, \ldots, \in \mathcal{F}$

Then $A_1^c, A_2^c, \ldots, \in \mathcal{F}$.

Thus $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}$.

Thus $\left[\bigcup_{i=1}^{\infty} A_i^c\right]^c = \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

43] A σ-algebra is a an algebra.

proof] Need to show $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$ Let $A_1 = A$, $A_2 = B$ and $A_i = \phi$ $i = 3, 4, \ldots$

$(A, A^c \in \mathcal{F} \Rightarrow A \cup A^c = X \in \mathcal{F} \Rightarrow X^c = \phi \in \mathcal{F})$

Thus $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 = A \cup B \in \mathcal{F}$.

44] Countable means finite or countably infinite for both sets and sequences.

ex] i) The largest σ-algebra consists of all subsets of X. ii) The smallest σ-algebra is $\mathcal{F} = \{\phi, X\}$.

For ii), $A_i \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i = \begin{cases} \phi & \text{if at least one } A_i = \phi \\ X & \text{if all } A_i = X \end{cases}$

$$\bigcup_{i=1}^{\infty} A_i = \begin{cases} \phi & \text{if all } A_i = \phi \\ X & \text{if at least one } A_i = X. \end{cases}$$

ex] Let Λ be a nonempty index set of sets $A_\lambda \subseteq X$.

prove $\left[\bigcup_{\lambda \in \Lambda} A_\lambda\right]^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$.

proof a) $x \in \left[\bigcup_{\lambda \in \Lambda} A_\lambda\right]^c$ iff $x \notin \bigcup_{\lambda \in \Lambda} A_\lambda$

iff $x \notin A_\lambda$ for any $\lambda \in \Lambda$ iff $x \in A_\lambda^c$ for all $\lambda \in \Lambda$ iff $x \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$.

proof b) If $x \in \left[\bigcup_{\lambda \in \Lambda} A_\lambda\right]^c$ then $x \notin \bigcup_{\lambda \in \Lambda} A_\lambda$.

Hence $x \notin A_\lambda$ for any $\lambda \in \Lambda$. $\therefore x \in A_\lambda^c \ \forall \lambda \in \Lambda$

$\therefore x \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$.

If $x \in \bigcap_{\lambda \in \Lambda} A_\lambda^c$, then $x \in A_\lambda^c$ for all $\lambda \in \Lambda$.

Hence $x \notin A_\lambda$ for any $\lambda \in \Lambda$. Thus

$x \notin \bigcup_{\lambda \in \Lambda} A_\lambda$. Thus $x \in \left[\bigcup_{\lambda \in \Lambda} A_\lambda\right]^c$. $\square$

45) A finite algebra $\mathcal{C} = \{B_1, \ldots, B_J\}$ is a σ-algebra.

Proof: Need to show $A_1, A_2, \ldots \in \mathcal{C}$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$.

An algebra is closed under finite unions. Since $\mathcal{C}$ is finite, $\mathcal{C}$ only has a finite number of sets, $B_1, \ldots, B_J$, say.

If $A_1, A_2 \ldots \in \mathcal{C}$, then only a finite number are distinct, $C_1, \ldots, C_k$ say where the C_i and k depend on $A_1, A_2, \ldots$.

Thus $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{K} C_i \in \mathcal{C}$. $\square$

ex) $\mathcal{C} = \{\phi, A, A^c, X\}$ is a σ-algebra ex) Let X be an infinite set.

Let $\mathcal{C}$ consist of finite and cofinite subsets of X where A is cofinite if A^c is finite.

a) $\mathcal{C}$ is an algebra

b) $\mathcal{C}$ is not a σ-algebra.

Proof) a) By def $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$

Let $A, B \in \mathcal{C}$

i) A finite, B finite $\Rightarrow$ $A \cup B$ (finite) $\in \mathcal{C}$

ii) A cofinite, B cofinite $\Rightarrow (A \cup B)^c = A^c \cap B^c \subseteq B^c$ (finite) $\in \mathcal{C}$

So $A \cup B$ is cofinite

iii) A finite B cofinite ((A cofinite B finite is similar. $(A \cup B)^c \subseteq A^c$ finite $\in \mathcal{C}$))

$(A \cup B)^c = A^c \cap B^c \subseteq B^c$ finite $\in \mathcal{C}$

Thus $A, B \in \mathcal{C} \Rightarrow A \cup B \in \mathcal{C}$

b) choose $A \subseteq X$ such that

A is countably infinite and A^c is infinite.

$A = \{a_1, a_2, \dots\} = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{a_i\}$

and A_i finite $\in \mathcal{C}$ but $A \notin \mathcal{C}$.

$\therefore$ $\mathcal{C}$ is not a σ algebra.

$\square$