$$(*)\begin{cases} \forall \varepsilon > 0 \ \exists N \ni \\ \left| \int_E f_n - \int_E f \right| < \varepsilon \quad \forall n > N. \end{cases}$$

Let $\varepsilon > 0$. Then $\left| \int_E f_n - \int_E f \right| \underset{\text{by linearity}}{=}$

$\left| \int_E (f_n - f) \right|$.

By Egoroff's th, $\exists A \ni$

$m(A) \le \frac{\varepsilon}{4M}$ and $f_n(x) \to f(x)$ uniformly

for $x \notin A$.

By uniform convergence on $E \setminus A$,

for $\varepsilon' = \frac{\varepsilon}{2(m(E)+1)}$, $\exists N_0 \ni$

$|f_n(x) - f(x)| < \varepsilon' \quad \forall n > N_0$ and

$\forall x \in E \setminus A$. $\therefore$

$$\left| \int_E f_n - \int_E f \right| \le \int_E |f_n - f|$$

$$\le \int_{E\setminus A} |f_n - f| + \int_A |f_n - f| \le$$

$$\leq \frac{\varepsilon}{2(m(E)+1)} m(E\setminus A) + 2M\, m(A)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } n \geq N_0.$$

Take $n = N_0$. Then $(*)$ is proved.

Thus $\lim_{n\to\infty} \int_E f_n = \int_E f$.

(used $E\setminus A \subseteq E$, so
$m(E\setminus A) \leq \underbrace{m(E)+1}_{\geq 1}$ and $\frac{m(E\setminus A)}{m(E)+1} \leq 1$).

avoids $\frac{0}{0}$

$\square$

Remark: In calculus, need $f_n \to f$ uniformly to get $(R)\int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} (R)\int_E f_n$.

31) uses pointwise convergence instead of uniform convergence, but needs $|f_n| \leq M$.

§4.3 L integral of nonnegative functions

32} * Def. Let $E \in \mathfrak{M}$, $f, h \in \mathcal{L}(E)$,

$f \geq 0$ and h bounded.

Then $\int_E f = \sup_{h \leq f} \int_E h$

where $m(\{x \in E : h(x) \neq 0\}) < \infty$.

33} Remarks:

i) need h bounded, measurable, and $m(\{x \in E : h(x) \neq 0\}) < \infty$ for RHS to be defined. (want h to vanish outside a set of finite measure)

ii) Now $m(E) = \infty$ is possible.

iii) $\int_E f = \infty$ is possible.

iv) $\chi_E \geq 0$ is nonnegative.

v) we can't replace sup by inf!

we can't define $\int_E f = \inf_{k \geq f} \int_E k$

k is bounded, measurable and

$m(\{x \in E : k \neq 0\}) < \infty$.

Reason: f need not be bounded

$\therefore$ $k \geq f$ need not be bounded.

vi) we can define

$$\int_E f = \sup_{h \leq f \text{ a.e.}} \int_E h$$

where h satisfies i).

vii) $\int_E f \geq 0$ ($h = 0 \leq f$ everywhere)

34)* If $m(E) = 0$ and $f \geq 0$ on E,

then $\int_E f = 0$. where $f \in \mathcal{S}(E)$.

proof} h is bounded and

$m(\{x \in E : h(x) \neq 0\}) < \infty$ means

$\int_E h$ is defined as in §4.2.

Thus $\int_E h \leq \int_E |h| \leq M_h \, m(E) = 0$

where $|h| \leq M_h$ on E.

Thus 0 is an upper bound on $\int_E h$

where h satisfies 33 i).

$\therefore$ $0 \leq \sup_{h \leq f} \int_E h \leq 0$ $\square$ sup is least upper bound

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35} ~~know~~ Def} Let $f \geq 0$ be measurable in $E \in \mathfrak{M}$.

If $\int_E f < \infty$ then f is integrable in E

36} prop 4.8. Let f and $g \in \mathcal{S}(E)$, $E \in \mathfrak{M}$.

Let $f \geq 0$, $g \geq 0$. Then

i) $\int_E cf = c\int_E f$ for $c > 0$.

ii) $\int_E f+g = \int_E f + \int_E g$ (restricted linearity $f,g \geq 0$)

iii) If $g \leq f$ a.e in E, then $\int_E g \leq \int_E f$.

proof

Let $B_E(h) = B_E$ denote $h \in \mathcal{S}(E)$, h is bounded in E and $m(\{x \in E : h(x) \neq 0\}) < \infty$.

i) $\int_E cf = \sup_{\substack{h \leq cf \\ B_E}} \int_E h = \sup_{\frac{h}{c} \leq f} c\int_E \frac{h}{c}$

$\frac{h}{c}$ is bdd $B_E(\frac{h}{c})$

$= c\int_E f$

$\frac{h}{c} = \tilde{h}$ since $c > 0$ ("we know how to integrate bdd functions")

$\square$

ii) $\int_E f+g = \sup_{\substack{h \leq f+g \\ B_E}} \int_E h$

Let $s = \min(f, h) \le f$.

s is bounded since $0 \le f$ and h is bounded.

If $h = 0$, then $s = 0$ since $f \ge 0$.

So if $s \ne 0$, then $h \ne 0$ and

$\{x : s \ne 0\} \subseteq \{x : h \ne 0\}$. Thus

$m(\{x : s \ne 0\}) \le m(\{x : h \ne 0\}) < \infty$.

Let $t(x) = h(x) - s(x)$.

Then $|t| \le |h| + |s|$ so t is bounded.

Now $t(x) = h(x) - \min(f(x), h(x))$

$= \max(h(x) - f(x), 0)$

$\le \max(g(x), 0) = g(x)$.

$\uparrow$

$h(x) \le f(x) + g(x)$

$$\therefore \int_E (f+g) = \sup_{\substack{h \\ B_E}} \int_E h \overset{h = s+t}{=} \sup_{\substack{h \\ B_E}} \int_E (s+t)$$

(If $h = 0$ then $t = 0$ so if $t \ne 0$ then $h \ne 0$
and $m(\{x : t \ne 0\}) \le m(\{x : h \ne 0\}) < \infty$.)

$\sup(a_n+b_n) \leq \sup a_n + \sup b_n$

$$\leq \sup_{\substack{s \leq f \\ B_E(s)}} \int_E s + \sup_{\substack{t \leq g \\ B_E(t)}} \int_E t$$

$$= \int_E f + \int_E g$$

$$\therefore \int_E (f+g) \leq \int_E f + \int_E g.$$

Now $\displaystyle \int_E f + \int_E g = \sup_{\substack{h \leq f \\ B_E(h)}} \int_E h + \sup_{\substack{k \leq g \\ B_E(k)}} \int_E k$

$$\leq \sup_{\substack{l \leq f+g \\ B_E(l)}} \int_E l = \int_E f+g.$$

$h+k \leq f+g$ ($\int_E h + \int_E k = \int_E (h+k) \leq \int_E (f+g)$

$\forall$ h, k as above. So $\displaystyle \sup_{\substack{h \\ B_E(h)}} \left(\int_E h + \int_E k\right) \leq \int_E (f+g)$

or $\int_E f + \int_E k \leq \int f+g$ $\forall$ k as above.

Thus $\displaystyle \int_E f + \sup_{\substack{k \\ B_E(k)}} \int_E k \leq \int_E f+g$

or $\int_E f + \int_E g \leq \int_E f+g$.

can have $\sup_n (a_n+b_n) \leq \sup_n a_n + \sup_n b_n$,

but we are taking sup over 2 different things, h then k.)

$\square$

iii) $\int_E g = \sup_{\substack{k \le g \\ B_E(k)}} \int_E k \le \sup_{\substack{h \le f \\ B_E(h)}} \int_E h = \int_E f$

$k \le g \Rightarrow k \le f$ so $h \le f$ has more functions

(true for the everywhere case so true for the ae case)

$\square$

37) * prop 4.13, Th. Let f and g be measurable in E and let $0 \le g \le f$ in E. If f is integrable, then g is integrable, and $\int_E (f-g) = \int_E f - \int_E g$.

proof) By 36 iii), $\int_E g \le \int_E f < \infty$

So g is integrable.

$\therefore \int_E f = \int_E (f-g+g) = \int_E \overbrace{(f-g)}^{\text{nonnegative}} + \int_E g$

So $\int_E (f-g) = \int_E f - \int_E g$

($\neq \infty - \infty$ so subtraction is allowed).

$\square$

38) **Know: Fatou's Lemma:**

If f_n is a sequence of nonnegative measurable functions in E and if $f_n(x) \to f(x)$ ae in E, then

$$\int_E f \le \underline{\lim} \underbrace{\int_E f_n}_{a_n}.$$

Note: This sequence may not converge, but $\underline{\lim}\, a_n$ always converges.

Proof) Let $F = \{x \in E : f_n(x) \not\to f(x)\}$.

Then $m(F) = 0$. Thus

$$\int_E f = \int_{E \setminus F} f + \underbrace{\int_F f}_{0} = \int_{E \setminus F} f.$$

WLOG assume that $f_n \to f$ in E (otherwise (ae), consider $E \setminus F$).

For any $h(x)$ with $h(x) \leq f(x)$,

$h(x)$ bounded msbl, and $m\{x: h(x) \neq 0\} < \infty$

$(B_E(h))$, Set $h_n(x) = \min(h(x), f_n(x))$.

Then h_n is bounded, $h_n \leq h \leq f$ (by def), $h_n \leq f_n$,

and $m(\{x: h_n(x) \neq 0\}) \leq m(\{x: h(x) \neq 0\}) < \infty$

(If $h = 0$ then $h_n = 0$ since $f_n \geq 0$
so if $h_n \neq 0$ then $h \neq 0$.)

Since $f_n(x) \to f(x) \geq h(x)$,

$h_n(x) = \min(h_n(x), f(x)) \to h(x)$.

By the Bounded Conv Th (BCT),

$$\int_E h(x) = \lim_{n \to \infty} \int h_n(x) = \varliminf_{n \to \infty} \int h_n(x).$$

($\underline{\lim} = \lim$ when $\lim$ exists)

$$\leq \varliminf_{n \to \infty} \int_E f_n. \quad (*)$$ Thus

$h_n \leq f_n$
$h_n \leq f$

($\varliminf a_n \leq \varliminf b_n$ if $a_n \leq b_n$)

$$\int_E \lim_{n\to\infty} f_n = \int_E f = \sup_{\substack{h\le f\\ B\epsilon(h)}} \int_E h \le \underbrace{\underline{\lim} \int_E f_n}_{\substack{\text{upper bound}\\ \text{on } \int_E h\\ \text{for any } h\le f}}$$

by (*):

$\square$

ex] If $f_n \ge 0$, if f_n is measurable over $E \in \mathfrak{M}$, if $f_n \to f$ a.e and if $\int_E f_n \le M < \infty \;\; \forall n$, then f is integrable and $\int_E f \le M$

proof] f_n measurable $\Rightarrow \underline{\lim}\, f_n = f$ is measurable.

By Fatou's lemma,

$$\int_E f = \int_E \lim_{n\to\infty} f_n \le \underline{\lim}_{n\to\infty} \int_E f_n \le M < \infty$$

So f is integrable.

$a_n = \int_E f_n \le M$ so the smallest limit point of $a_n \le M$

$\square$

39] p87 Know **Monotone Convergence Theorem (MCT)**:

Let $0 \le f_n \le f_{n+1}$, and f_n measurable in $E \in \mathfrak{M}$

If $f_n \to f$ ae in E, then $\int_E f_n \to \int_E f$.

Remark: $0 \le \underbrace{f_n \uparrow f}_{\text{monotone convergence}}$ ae on E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f = \int_E \lim_{n\to\infty} f_n.$$

Proof} <u>know</u> By Fatou's lemma,

$$\int_E f \le \varliminf_{n\to\infty} \int_E f_n.$$

Since $f_n \uparrow f$, for any n

$f_n(x) \le f(x)$ ae in E.

Thus $\int_E f_n \le \int_E f \quad \forall n$

So $\int_E f \overset{\text{Fatou}}{\le} \varliminf \int_E f_n \le \varlimsup \int_E f_n \le \int_E f.$

$$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$$

$\square$

40} <u>know</u> Let $u_i \ge 0$ be measurable over $E \in \mathfrak{M}$.

Let $f(x) = \sum_{i=1}^{\infty} u_i(x)$, x in E. Then

$$\int_E f(x)\,dx = \int_E \sum_{i=1}^{\infty} u_i(x)\,dx = \sum_{i=1}^{\infty} \int_E u_i(x)\,dx.$$

Proof} know Let $f_n(x) = \sum_{i=1}^{n} u_i(x)$.

Then $f_n \ge 0$, $f_n \uparrow f$, and f_n is measurable.

$\therefore$ by the MCT,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Now $\lim_{n\to\infty} \int_E f_n(x)\,dx = \lim_{n\to\infty} \int_E \sum_{i=1}^{n} u_i(x)\,dx$

$$\underset{\substack{\text{restricted}\\\text{linearity}\\\text{for finite sum}}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i(x)\,dx \underset{(*)}{=} \sum_{n=1}^{\infty} \int_E u_i(x)\,dx.$$

$\square$

4B Remarks: i) $(*)$ holds because

$\int_E u_i(x)\,dx$ is nonnegative.

$\therefore \sum_{i=1}^{n} \int_E u_i(x)\,dx$ is monotone increasing

and has a limit (and we write

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i(x)\,dx = \sum_{i=1}^{\infty} \int_E u_i(x)\,dx.$$

ii) In general, $\sum_{i=1}^{n} \int_E u_i(x)\,dx$

does not have a limit, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_E u_i(x)\,dx \neq \sum_{i=1}^{\infty} \int_E u_i(x)\,dx.$$

iii) 40) gives conditions under which the infinite sum and integral operators can be interchanged.

(v) Riemann integral needs $f_n \to f$ uniformly (instead of pointwise).

42) * Prop 4.12 Th Let $f \geq 0$ be measurable over E. Let $E_i \in \mathcal{M}$ be disjoint with $E = \bigcup_{i=1}^{\infty} E_i$.

Then $\displaystyle\int_{\bigcup_{i=1}^{\infty} E_i} f = \sum_{i=1}^{\infty} \int_{E_i} f.$

Proof) $\displaystyle\sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx = \sum_{i=1}^{\infty} \int_E f(x)\,\chi_{E_i}(x)\,dx$

Since $E_i \subseteq E \Rightarrow \chi_E(x)\chi_{E_i}(x) = \chi_{E_i}(x)$.

Let $u_i(x) = f(x)\,\chi_{E_i}(x)$ for $x \in E$.

Then $\displaystyle\sum_{i=1}^{\infty} u_i(x) = \sum_{i=1}^{\infty} f(x)\,\chi_{E_i}(x) = f(x) \underbrace{\sum_{i=1}^{\infty} \chi_{E_i}(x)}_{\chi_E(x)}$