$= f(x)\chi_E(x) = f(x)$ for x in E.

By 40), $\int_E f(x)\,dx = \int_E f(x)\chi_E(x)\,dx$

$$= \int_E \sum_{i=1}^{\infty} v_i(x)\,dx \overset{40)}{=} \sum_{i=1}^{\infty} \int_E v_i(x)\,dx$$

$$= \underbrace{\sum_{i=1}^{\infty}}_{\text{def of } v_i} \int_E f(x)\chi_{E_i}(x)\,dx \underset{\text{def of } \int f\chi_{E_i}}{=} \sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx.$$

$\square$

43) Remarks i) $\int_{A\cup B} f = \int_A f + \int_B f$, $A \cap B = \emptyset$.

So by induction, if the E_i are disjoint,

then $\displaystyle\int_{\bigcup_{i=1}^{n} E_i} f = \sum_{i=1}^{n} \int_{E_i} f$. (*)

But we did not have theory saying

that $\displaystyle\lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} E_i} f = \int_{\bigcup_{i=1}^{\infty} E_i} f = \int_E f$.

(can't take limits in (*) unless it is known that the limit exists.

44) A standard techinique is to approximate f by bounded

functions f_n. In the next proof, $|f_n| \le n$ is bounded, and $f_n \uparrow f$.

45} ("continuity of integral"): Prop 4.14

Th} Let $f \ge 0$ and integrable over $E \in \mathfrak{M}$. Then for any $\varepsilon > 0$, $\exists\, \delta > 0 \ni$ for <u>any set</u> $A \subseteq E$ <u>with $m(A) < \delta$</u>,

$$\int_A f(x)\,dx \le \varepsilon.$$

Proof} Let the truncated functions

$$f_n(x) = \begin{cases} n, & \{x \in E : f(x) > n\} \\ f(x), & \{x \in E : f(x) \le n\}. \end{cases}$$

Then $0 \le f_n = |f_n| \le n$ is bdd and $f_n \le f_{n+1}$.

i) f_n is measurable since $f_n = \min(f, n)$.

ii) $f_n \uparrow f$.

iii) by MCT, $\lim_{n \to \infty} \int_E f_n = \int_E f$.

By def of the limit, for the ε in the theorem,

$\exists N_0 \ni 0 \le \int_E f - \int_E f_n < \frac{\varepsilon}{2} \quad \forall n \ge N_0.$

$\therefore \int_E (f - f_{N_0}) < \frac{\varepsilon}{2}.$

Hence $\int_A f \overset{(*)}{=} \int_A (f - f_{N_0}) + \int_A \underbrace{f_{N_0}}_{|f_{N_0}| \le N_0}$

$\le \frac{\varepsilon}{2} + N_0\, m(A).$

Take $\delta = \dfrac{\varepsilon}{2(N_0+1)}$ so $m(A) < \delta$.

Then $\int_A f \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

for any $A \subseteq E$ with $m(A) < \delta$.

(Note that (*) holds since if $A \subseteq E$ and $f \ge 0$ then

$\int_E f = \int_A f + \underbrace{\int_{E \setminus A} f}_{\ge 0} \Rightarrow \int_A f \le \int_E f.$)

$\square$

§4.4 General Lebesgue Integral

(f need not be bounded f need not be nonnegative.)

46} know Def. If $f: D \to [-\infty, \infty]$,

then the <u>positive part</u> $f^+ = f\, I(f \geq 0) = \max(f, 0)$

and the <u>negative part</u> $f^- = -f\, I(f \leq 0) = \max(-f, 0)$
$= -\min(f, 0)$.

47} know f^+ and f^- are measurable if f is.

$f = f^+ - f^-$

$|f| = f^+ + f^-$ is nonnegative.
f^+ and f^- are nonnegative.

48} know Def) A measurable function f is <u>L integrable over a measurable set E</u> if f^+ and f^- are both L integrable over E.

49) know Th} Let <u>f be a measurable function over $E \in \mathcal{F}_m$.</u>

a) Then f is L integrable iff $|f|$ is L integrable.

b) If f is L. integrable,

then $\left|\int_E f\right| \leq \int_E |f| < \infty$.

proof] a) know if f is L. integrable, then f^+ and f^- are L. integrable, and

$$\int_E |f| = \int_E (f^+ + f^-) = \underbrace{\int_E f^+}_{<\infty} + \underbrace{\int_E f^-}_{<\infty} < \infty$$

$\therefore |f|$ is L. integrable.

Suppose $|f|$ is L. integrable. Then

$$\int_E |f| = \underbrace{\int_E f^+}_{\in[0,\infty)} + \underbrace{\int_E f^-}_{\in[0,\infty)} < \infty.$$

both integrals need to be finite since both are nonneg.

$\therefore f^+$ and f^- are L. integrable

$\therefore f$ is L. integrable

$\square$

b) see HW 9

It is good to know (counter) examples.

ex) $f = \chi_{(0,\infty)}$ is <u>not</u> L. integrable since

$$\int \chi_{(0,\infty)} = \int |\chi_{(0,\infty)}| = m((0,\infty)) = \infty$$

50} Know Def If the measurable function f is L. integrable over measurable set E, then the L. integral is $\int_E f = \int_E f^+ - \int_E f^-$.

ex} know Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ -1 & x \notin \mathbb{Q} \cap [0,1]. \end{cases}$

Then f is not Riemann integrable, but $|f| \equiv 1$ is Riemann integrable with

$$R \int_0^1 |f| = \int_0^1 dx = x\Big|_0^1 = 1$$

Note that measurable f is L. integrable iff $|f|$ is L. integrable. This result is not true for the Riemann integral by the above ex.

51} Th. Unless otherwise stated, assume that f, g and f_i are measurable and L integrable, and that all indicated sets are measurable.

a) $\int_E cf = c\int_E f$

b) $\int_E c = c\, m(E)$ (even if $m(E) = \infty$, in which case χ_E is not L measurable)

c) If $m(E) = 0$, then $\int_E f = 0$

d) If $E = \bigcup_{i=1}^{n} E_i$ where the E_i are disjoint,

then $\int_E f = \sum_{i=1}^{n} \int_{E_i} f$.

e) (linearity) $\int_E \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \int_E f_i$

f) (monotonicity) If $f \le g$ ae on E,

then $\int_E f \le \int_E g$.

g) $\left| \int_E f \right| \le \int_E |f|$

h) If $f = g$ ae on E, then $\int_E f = \int_E g$

i) If $f \ge 0$ ae on E and $\int_E f = 0$, then

$f(x) = 0$ ae on E.

j) If $\int_E f$ exists, then f is finite ae on E.

k) If $\int_E f$ exists and $A \subseteq E$, then

$\int_A f$ exists and $\int_A |f| \le \int_E |f|$.

know a) – f)

proofs) a) $\int_E cf = \int_E (cf)^+ - \int_E (cf)^-$

$= c \int_E f$ if $c \ge 0$

If $g \le 0$, then $\int_E g = -\int_E |g|$.

Let $c < 0$.

Now $(cf)^+ = |cf^-|$ and $(cf)^- = |cf^+| = -cf^+$

So $\int_E (cf)^+ - \int_E (cf)^- =$

$$-c\int_E f^- + c\int_E f^+ = \int_E f.$$

b) follows from 26) since χ_E is bdd, msbl.

c) $\int_E f = \int_E f^+ - \int_E f^- = 0 - 0$ (from 34)

d) $\int_E f = \int_E f^+ - \int_E f^- \underset{\text{b/43)}}{=} \int_{\bigcup_{i=1}^{n} E_i} f^+ - \int_{\bigcup_{i=1}^{n} E_i} f^-$

$$= \int_{\bigcup_{i=1}^{n} E_i} f$$

<u>or</u> $\int_{A\cup B} f \underset{A\cap B=\emptyset}{=} \int f(\chi_A + \chi_B) = \int_A f + \int_B f$

and then use induction.

e) Let $h^{\pm} = h^+$ or h^-. Then

$\int_E (f+g)^{\pm} \leq \int_E \left((f+g)^+ + (f+g)^-\right) = \int_E |f+g| \leq \int_E |f| + \int_E |g|$
$< \infty$. $\therefore$ $(f+g)^{\pm}$ are integrable. $\therefore$ $f+g$ is integrable.

Let $h = f+g$. Then $\underbrace{h^+ - h^-}_{f+g} = \underbrace{f^+ - f^-}_{f} + \underbrace{g^+ - g^-}_{g}$ or

$h^+ + f^- + g^- = h^- + f^+ + g^+$ (everything ≥ 0 so)

$\int_E h^+ + \int_E f^- + \int_E g^- = \int_E h^- + \int_E f^+ + \int_E g^+$ or

$\int_E h^+ - \int_E h^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$

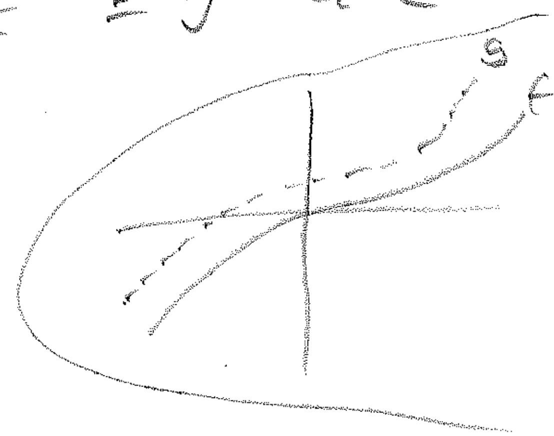
or $\int_E h \leq \int_E (f+g) = \int_E f + \int_E g$

Then by induction $\int_E \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} \int_E f_i$.

e) If $f \leq g$ ae, then

$f = f^+ - f^- \leq g^+ - g^-$ ae

$\therefore f^+ \leq g^+$ ae and $\underbrace{-f^- \leq -g^- \text{ ae}}_{f^- \geq g^- \text{ ae}}$



$\therefore \int_E f = \int_E f^+ - \int_E f^-$

$\leq \int_E g^+ - \int_E g^- = \int_E g$

$\int_E f^+ \leq \int_E g^+$ and $\int_E f^- \geq \int_E g^-$ by result for nonneg fns

$-\int_E f^- \leq -\int_E g^-$

$\square$

text omits this word

523 Know Lebesgue's Dominated Convergence Theorem (LDCT):

Prop 4.16

Let g be integrable over $E \in \mathfrak{M}$ and let f_n be a sequence of measurable functions over E such that $|f_n(x)| \le g(x)$ on E and $f(x) = \lim_{n\to\infty} f_n(x)$ a.e. on E.

Then $\int_E f = \int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_E f_n$.

Proof} Let $h_n(x) = g(x) - f_n(x)$. Thus know $h_n(x) \ge 0$ and $h_n \to g - f$ a.e. on E.
Since $|f| \le g$, f is integrable.

By Fatou's lemma,

$$\int_E (g-f) \le \underline{\lim} \int_E \overbrace{(g-f_n)}^{h_n} \therefore$$

$$\int_E g - \int_E f \le \int_E g - \overline{\lim} \int_E f_n$$

$$\underline{\lim}(-a_n) = -\overline{\lim}(a_n) \qquad (\text{no } \infty - \infty)$$

Thus $\overline{\lim} \int_E f_n \le \int_E f$. (*)

Let $k_n = g + f_n \ge 0$. By Fatou's lemma

$$\int_E (g+f) \le \underline{\lim} \int_E (g+f_n), \text{ or}$$

$$\int_E g + \int_E f \le \int_E g + \underline{\lim} \int_E f_n.$$

Hence $\int_E f \le \underline{\lim} \int_E f_n$.

$\therefore \int_E f \le \underline{\lim} \int_E f_n \le \overline{\lim} \int_E f_n \overset{(*)}{\le} \int_E f$

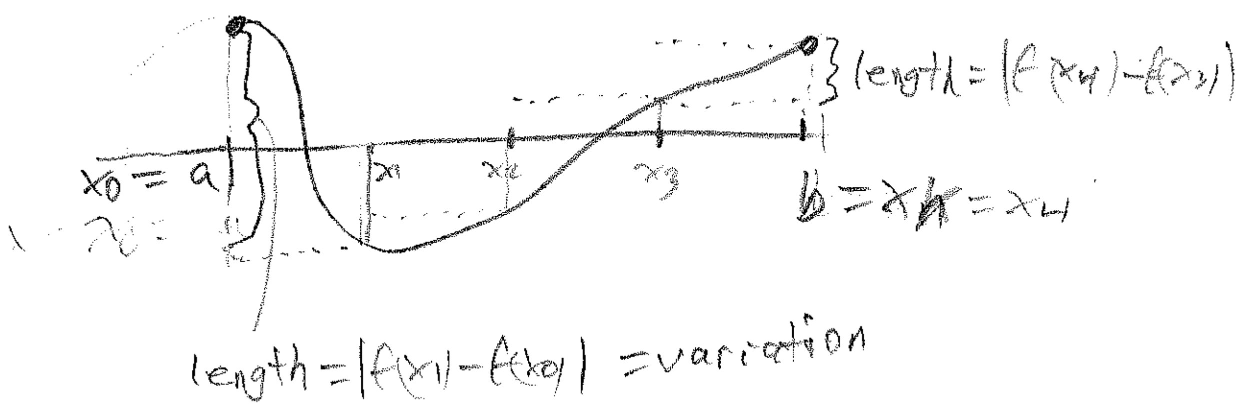
$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$.

$\square$

ex) Does $\int_E f\,dx = 0 \Rightarrow f = 0$ ae in E?

Soln) No, true if $f \ge 0$.

ch5 §5.2 Functions of Bounded Variation



1) Def. Let function $f: [a,b] \to \mathbb{R}$ and let $\pi = a = x_0 < x_1 < x_2 < \dots < x_k = b$ be a partition of $[a,b]$.

Let $p = {}^{+}V_a^b(f,\pi) = \underbrace{\sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^{+}}_{\text{positive part}}$.

Let $n = {}^{-}V_a^b(f,\pi) = \underbrace{\sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^{-}}_{\text{"negative part", nonnegative}}$

Let $t = V_a^b(f,\pi) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$.

Let $\underline{P} = {}^{+}V_a^b(f) = \sup\{{}^{+}V_a^b(f,\pi),\ \pi \text{ a partition of } [a,b]\}$

Let $N = {}^{-}V_a^b(f) = \sup\{{}^{-}V_a^b(f,\pi),\ \pi \text{ a partition of } [a,b]\}$

Let $T = V_a^b(f) = \sup\{V_a^b(f,\pi),\ \pi \text{ a partition of } [a,b]\}$.

Then $T = V_a^b(f) =$ <u>total variation</u> of f over $[a,b]$.

$\underline{P} = {}^{+}V_a^b(f) =$ <u>positive variation</u> of f over $[a,b]$,

$N = {}^{-}V_a^b(f) =$ <u>negative variation</u> of f over $[a,b]$,

Then f is of <u>bounded variation</u> over $[a,b]$, <u>written f is BV</u>, if $T = V_a^b(f) < \infty$.

Remark! Here $r^{+} = \begin{cases} r & r \geq 0 \\ 0 & r \leq 0 \end{cases}$ and $r^{-} = |r| - r^{+}$

$r^{-} = \begin{cases} -r, & r < 0 \\ 0 & r \geq 0. \end{cases}$ $\qquad r = r^{+} - r^{-}$