Sol 7)

2) P 103 Lemma 5.4. Suppose $f : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$. Then

total variation = positive variation + negative variation

$$T = V_a^b(f) = {}^+V_a^b(f) + {}^-V_a^b(f) = P + N \quad (a)$$

and $$f(b) - f(a) = {}^+V_a^b(f) - {}^-V_a^b(f) = P - N \quad (b)$$

proof) First prove (b). For any partition

π: $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ of $[a,b]$,

$$f(b) - f(a) \underset{\text{telescoping}}{=} \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})].$$

$$\underset{x = x^+ - x^-}{=} \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+ - \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$= {}^+V_a^b(f,\pi) - {}^-V_a^b(f,\pi).$$

$\therefore$ $$f(b) - f(a) + {}^-V_a^b(f,\pi) = {}^+V_a^b(f,\pi) \quad (*)$$

is true over all partitions.

In $(*)$, take $\sup_\pi$ over all partitions

$$\sup_\pi [f(b) - f(a) + {}^-V_a^b(f,\pi)] = \sup_\pi {}^+V_a^b(f,\pi)$$

or $$f(b) - f(a) + {}^-V_a^b(f) = {}^+V_a^b f$$

(Needed $(*)$ because $\sup(a_\pi - b_\pi) = \sup a_\pi - \inf a_\pi$)

Now f is of BV, so

$$ {}^{\pm}V_a^b(f) = \sup_{\pi} {}^{\pm}V_a^b(f,\pi) \leq \sup_{\pi} V_a^b(f,\pi) $$

$$ = V_a^b(f) \underset{\substack{\uparrow \\ BV}}{<} \infty. $$

$|x| = x^+ + x^-$, $x^+ \leq |x|$, $x^- \leq |x|$

so $\sup x^+ \leq \sup|x|$, $\sup x^- \leq \sup|x|$

So ${}^{\pm}V_a^b(f) < \infty$ and (b) holds.

$\sup_{\pi}(a_\pi + b_\pi) \leq \sup_{\pi} a_\pi + \sup_{\pi} b_\pi$ so holds (**)

For (a), use $V_a^b(f,\pi) \underset{\uparrow}{=} {}^+V_a^b(f,\pi) + {}^-V_a^b(f,\pi)$

(π is fixed): using $|x| = x^+ + x^-$

solve for $-V_a^b(f,\pi)$ in (*)

$$ = {}^+V_a^b(f,\pi) + {}^+V_a^b(f) - [f(b) - f(a)] \text{ by } (*) $$

$$ = 2\,{}^+V_a^b(f,\pi) - [f(b) - f(a)] \qquad (**) $$

Take $\sup_{\pi}$ in (**) to get

$$ V_a^b(f) = 2\,{}^+V_a^b(f) - [f(b) - f(a)] $$

$$ \underset{\substack{\uparrow \\ \text{by b)}}}{=} 2\,{}^+V_a^b(f) - [{}^+V_a^b(f) - {}^-V_a^b(f)] $$

$$ = {}^+V_a^b(f) + {}^-V_a^b(f) $$

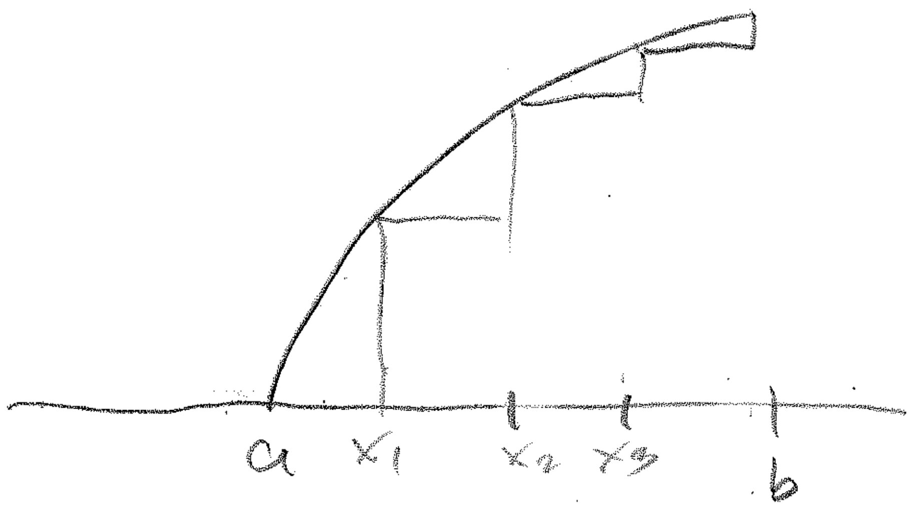
and (a) holds.

$\square$

501 71.5

ex] For a monotone (nondecreasing or nonincreasing function) h, $\overset{\text{over } [a,b]}{T} = |h(b) - h(a)|$. So h is BV.

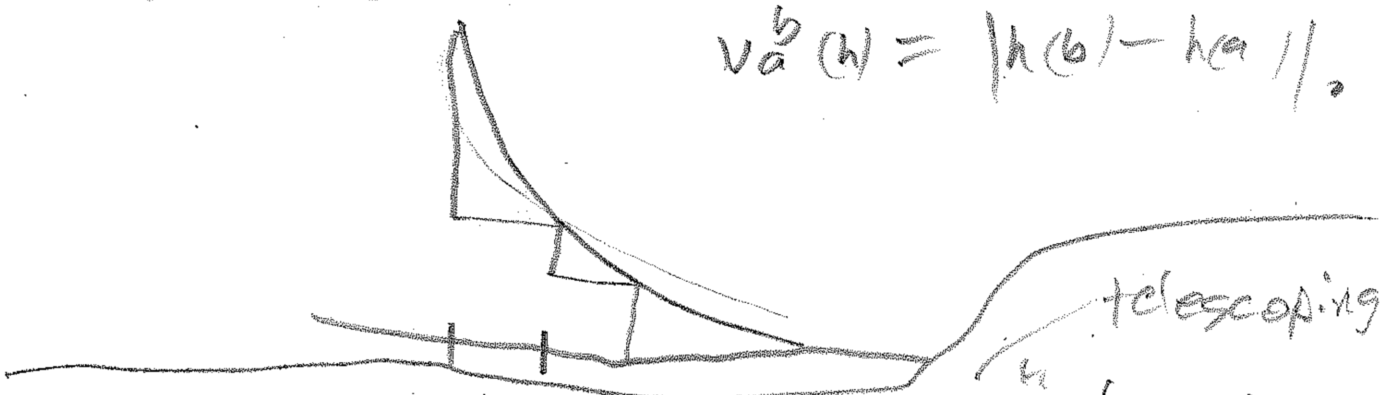
proof sketch.



For any partition π, we get a monotone stepfunction for $h(x_i) - h(x_{i-1})$ with lengths that sum to $|h(b) - h(a)|$.

So $V_a^b(h,\pi) = |h(b) - h(a)|$ $\forall \pi$, so

$$V_a^b(h) = |h(b) - h(a)|.$$



For increasing h, and any partition π

$$\sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| = \sum_{i=1}^{k} \left(h(x_i) - h(x_{i-1})\right)$$

$$= h(x_k) - h(x_0) = h(b) - h(a)$$

3) Th: A function $f:[a,b]\to\mathbb{R}$ is of BV iff f is the difference of two nondecreasing real-valued functions.

4) Th If f is of BV on $[a,b]$, then $f'(x)$ exists ae in $[a,b]$

5) a) $\underline{P} \leq T$
$N \leq T$
$T \leq \underline{P} + N$

b) $\underline{P} \geq 0$, $N \geq 0$, $T \geq 0$

6) p99 $f:[a,b]\to\mathbb{R}$ is <u>nondecreasing</u> if $f(x) \leq f(y)$ for $x \leq y$ and <u>nonincreasing</u> if $f(x) \geq f(y)$ for $x \leq y$.
If f is nonincreasing or nondecreasing, then f is a monotone function.
f is increasing if $f(x) < f(y)$ for $x < y$
f is decreasing if $f(x) > f(y)$ for $x < y$.

Some texts use increasing and strictly increasing.

§5.4

7] Know Def. A function $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ if given $\varepsilon>0$, $\exists\ \delta>0\ \ni$

$$\sum_{i=1}^{n} |f(x_i')-f(x_i)|<\varepsilon \quad \text{for}$$

every finite collection $\{(x_i,x_i')\}$ of nonoverlapping intervals with $\sum_{i=1}^{n} |x_i'-x_i|<\delta$.

8] Remarks i) Could use intervals (x_i, x_i+h_i) with $x_i'=x_i+h_i$

ii) Every absolutely continuous function is continuous.

iii) If f and g are absolutely continuous then so are $f-g$, $f+g$, $f(x)g(x)$, $f(x)/g(x)$ where $g(x)\neq 0$.

iv) If f is absolutely continuous on $[a,b]$ then f is of bounded variation on $[a,b]$.

v) If f is absolutely continuous, then

Sol 73

F has a derivative F' ae on $[a,b]$.

9) Def. If f is a L integrable function on $[a,b]$ then its <u>indefinite integral</u> is

$$F(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a,b].$$

10) Th A function F is an indefinite integral iff F is absolutely continuous.

11) Remark: So F is continuous and of BV. (by 8))

12) Th: Suppose $F(x) = \int_a^x f(t)\,dt + F(a)$.

If either a) f is bounded and measurable on $[a,b]$ or b) f is integrable on $[a,b]$,

then $F'(x) = f(x)$ ae in $[a,b]$.

§6| L^p spaces

1} Def The space of all functions $f(x)$ for which $|f(x)|^p$, $p \ge 1$ is L, integrable on $[a,b]$, ie

$$\int_a^b |f(x)|^p \, dx < \infty,$$

is the <u>L^p space</u>: $L^p = L^p[a,b]$.

2} The $L^2[a,b]$ space is the <u>Hilbert space</u> with functions $f \ni \int_a^b |f(x)|^2 dx = \int_a^b [f(x)]^2 dx < \infty$.

Functions belonging to the Hilbert space are often said to be square integrable.

3} The $L^1[a,b]$ space consists of the L, integrable functions on $[a,b]$ since $\int_a^b |f| < \infty$. L^1 is often denoted as L.

4) Cauchy Schwarz Inequality:

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \sqrt{\int_a^b |f(x)|^2\,dx}\,\sqrt{\int_a^b |g(x)|^2\,dx}$$

where $f, g \in L^2$.

$\uparrow$ $p=2, q=2$

5) Holder's Inequality:

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, $f \in L^p$ and $g \in L^q$.

$p>1$ forces $q>1$

6) Minkowski's Inequality:

$$\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p\,dx\right)^{\frac{1}{p}}$$

where $p \ge 1$, $f, g \in L^p$.

7) Th. If $1 \le p \le q$ then $L^q \subseteq L^p \subseteq L^1$

So if $f \in L^q$ then $f \in L^p$ and $f \in L^1$.

8) Th. If $f \in L^1$, $g \in L^p$, and $|f| \le |g|$, then $f \in L^p$.

9) Def. Let $f_n, f \in L^p[a,b]$.

If $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|^p\,dx = 0$

then f_n converges in mean to f in the space L^p.

10] Let $\|f\|_p = \left[\int_a^b |f|^p\right]^{1/p}$.

Ch 11

§11.1 Measure spaces

1]* Def A measurable space is $(X, \mathcal{F})$ consisting of a set X and a σ-algebra $\mathcal{F}$.

A set $A \subseteq X$ is measurable (wrt $\mathcal{F}$) if $A \in \mathcal{F}$.

Remark: A is also called a measurable set.

2] Often $\mathcal{F} = \mathcal{B}(X)$ = Borel σ-algebra on X, the universal set.

3] Know Def: A measure μ on measurable space $(X, \mathcal{F})$ is i) a nonnegative set function defined for all subsets of $\mathcal{F}$

ii) $\mu(\emptyset) = 0$

iii) (countable additivity): Let $E_1, E_2 \ldots$ be disjoint measurable sets

Then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$.

4) Know A measure space $(X, \mathfrak{F}, \mu)$ consists of a measurable space $(X, \mathfrak{F})$ and a measure μ defined on $(X, \mathfrak{F})$.

ex) i) $(X, \mathfrak{F}_m, m)$ $\mu = m =$ L. measure

ii) $(X, \mathcal{B}(X), m)$

iii) $(X, \mathcal{B}(X), P)$ where probability measure $\mu = P$ satisfies $P(X) = 1$.
(Math 581)

6) Know more properties of a measure $\mu : \mathfrak{F} \to [0, \mu(X)]$ where $0 < \mu(X) \leq \infty$ depends on μ.

iv) (monotonicity): If $A \subseteq B$ are measurable, then $\mu(A) \leq \mu(B)$.

v) (countable subadditivity): Let $E_1, E_2, \ldots$ be measurable, then $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

vi) (finite subadditivity): Let $E_1, \dots, E_N$ be measurable. Then $\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mu(E_i)$.

vii) (finite additivity): Let $E_1, \dots, E_N$ be disjoint measurable sets. Then $\mu\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} \mu(E_i)$.

viii) If the $E_i \in \mathfrak{F}$ (are measurable), $\mu(E_1) < \infty$, $E_{i+1} \subseteq E_i$, then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Some proofs iv) $B = A \cup (B-A)$ disjoint

$$\text{So } \mu(B) \underset{\substack{\uparrow \\ \text{by vii)}}}{=} \mu(A) + \mu(B-A) \geq \mu(A)$$

vi) Take $E_{N+1}, \dots = \phi$

$$\text{Then } \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{N} E_i\right) \underset{\substack{\uparrow \\ \text{countable additivity}}}{=} \sum_{i=1}^{\infty} \mu(E_i)$$

$$= \sum_{i=1}^{N} \mu(E_i)$$

viii) use the same proof as
for m but replace m by μ
see notes 39 34) prop 3.14.

7) Remark i) m used m^* to get results
including countable additivity.
μ assumes countable additivity.
ii) proofs that worked for m without m^*
often work for μ. See HW 10.

§ 11.2

8) Measurable functions are almost
the same as for L measure m.
Just replace m by μ and
$\mathcal{F}_m$ by $\mathcal{F}$.
See exam 3 review 60) and 61).

§ 11.3
9) Integration with respect to μ
is nearly the same as

L. integration if μ is a complete measure: $B \in \mathcal{F}$, $A \subseteq B$, $\mu(B) = 0 \Rightarrow A \in \mathcal{F}$.

a) Get results for nonnegative simple functions (instead of bounded functions).

b) Get results for measurable nonnegative functions.

c) Get results for general f using $\int_E f = \int_E f^+ - \int_E f^-$.

d) $\int_E f = \int_E f \, d\mu$ so

$$\int \chi_A \, d\mu = \mu(A) \quad , E, A \in \mathcal{F}.$$

e) Fatou's lemma, MCT, LDCT are nearly the same.

f) The integrals can be <u>very different</u> from a Riemann integral

Ex) probability $\mu = P$ $\quad \int_A dP = \int \chi_A dP = P(A)$

$$\int h \, dP = E[h(Y)] = \underbrace{\int_{-\infty}^{\infty} h(y) p(y) \, dy}_{\text{Riemann}} \quad \text{pdf } p(y)$$

$$\int h \, dP = \sum_{y : p(y) > 0} h(y) p(y) \quad \text{pmf } p(y)$$

10] Def. μ is a <u>complete measure</u>

if $B \in \mathcal{F}$, $A \subseteq B$ and $\mu(B) = 0$

$\Rightarrow A \in \mathcal{F}$.

ex] L measure on $\mathcal{F}_M$ is a complete measure

L measure on $\mathcal{B}[0,1] \subseteq \mathcal{F}_M[0,1]$

is not a complete measure.

11] Assume μ is a complete measure unless ~~stated~~ otherwise,

Assume measurable sets and measurable functions are wrt μ and $\mathcal{F}$ for $(X, \mathcal{F}, \mu)$.

12] Fatou's Lemma: Let f_n be a sequence of nonnegative measurable functions with $f_n \to f$ ae on $E \in \mathcal{F}$.

Then $\int_E f \, d\mu \leq \underline{\lim} \int_E f_n \, d\mu$.

13) MCT: Let f_n be a sequence of nonnegative measurable functions with $f_n \to f$ ae on $E \in \mathcal{F}$ and

suppose $f_n \le f$ $\forall n$. Then

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu.$$

14) Suppose f and g are nonnegative measurable functions and a and b nonnegative constants.

i) (restricted linearity):

$$\int_E (af + bg) \, d\mu = a \int_E f \, d\mu + b \int_E g \, d\mu$$

ii) $\int_E f \, d\mu \ge 0$

iii) nonnegative measurable f is integrable over $E \in \mathcal{F}$ if $\int_E f \, d\mu < \infty$.

15) Def An arbitrary measurable function f is integrable if both f^+ and f^- are integrable. Then the integral $\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$.

16) Suppose f, g, f_i are integrable and $E \in \mathcal{F}$ (is a measurable set).

i) $\int_E (c_1 f + c_2 g)\, d\mu = c_1 \int_E f\, d\mu + c_2 \int_E g\, d\mu$

ii) linearity: so $\int_E \sum_{i=1}^{n} a_i f_i \, d\mu$

$$= \sum_{i=1}^{n} a_i \int_E f_i \, d\mu$$

iii) If $|h| \le |f|$ and h is measurable, then h is integrable

iv) monotonicity: If $f \ge g$ ae, then $\int_E f\, d\mu \ge \int_E g\, d\mu$.

17) LDCT: Let g be integrable over $E \in \mathcal{F}$. Let f_n be a sequence of measurable functions such that $|f_n(x)| \le g(x)$ on E and $f_n(x) \to f(x)$ ae on E. Then

$$\int_E f\, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu$$

§11.6 Radon Nikodym Theorem

p266

18} Let f_n be a sequence of nonnegative functions. Then

$$\int_E \sum_{i=1}^{\infty} f_i \, d\mu = \sum_{i=1}^{\infty} \int_E f_i \, d\mu.$$

19} Def Let $(X, \mathcal{F}, \overset{\text{mu}}{\mu})$ and $(X, \mathcal{F}, \overset{\text{nu}}{\nu})$ be two measure spaces with the same X and $\mathcal{F}$.

i) Then μ and ν are <u>mutually singular</u> if there are disjoint sets $A, B \in \mathcal{F} \ni X = A \cup B$ and $\nu(A) = \mu(B) = 0$.

ii) A measure ν is <u>absolutely continuous</u> wrt μ, written $\nu \ll \mu$, if $\nu(A) = 0$ for any set $A \in \mathcal{F}$ for which $\mu(A) = 0$.