20] Th. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f be a nonnegative measurable function on X. For each $E \in \mathcal{F}$, define $\nu(E) = \int_E f \, d\mu$.

Then ν is a measure on $(X, \mathcal{F})$.

proof) i) $\nu(\phi) = \int_\phi f \, d\mu = \int \underbrace{f \chi_\phi}_{0} \, d\mu$

$= \int 0 \, d\mu = 0 \, \mu(X) = 0.$ (by def)

ii) ν is a nonnegative set function on $\mathcal{F}$ since μ is and since $f \geq 0$.

iii) Let $E_1, E_2, \ldots \in \mathcal{F}$ be disjoint.

Then $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \int_{\bigcup_{i=1}^{\infty} E_i} f \, d\mu$

$= \int \sum_{i=1}^{\infty} \underbrace{f \chi_{E_i}}_{g_i} \, d\mu \underset{\text{by 18]}}{=} \sum_{i=1}^{\infty} \int f \chi_{E_i} \, d\mu$

$= \sum_{i=1}^{\infty} \int_{E_i} f \, d\mu = \sum_{i=1}^{\infty} \nu(E_i).$

$\therefore$ ν is countably additive and

ν is a measure $\square$

ex] $P(A) = \int_A \underbrace{f(x)}_{\text{pdf}} \, dx$ with $\mu = m$ and $\nu = P$.

21) p256 Def. A measure μ is <u>σ-finite</u> if there is a sequence of measurable sets $X_i \in \mathcal{F}$ such that $X = \bigcup_{i=1}^{\infty} X_i$ and $\mu(X_i) < \infty$.

22) Fact on 21) the sequence X_i can always be taken to be disjoint.

23) <u>KNOW</u> Th. 11.23. Radon-Nikodym Theorem:

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, and let ν be a measure defined on $\mathcal{F}$ which is absolutely continuous wrt μ. Then there is a nonnegative measurable function f such that for each $E \in \mathcal{F}$, $\nu(E) = \int_E f \, d\mu$.

The function f is "unique" in that if g is any measurable function with this property, then $f = g$ ae μ.

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24] The function f in 23] is called the Radon-Nikodym derivative of ν wrt μ and is sometimes denoted by $\frac{d\nu}{d\mu}$. so $\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu$
$= \int_E d\nu = \nu(E)$

ex] 23) can be used to show that a prob measure $\nu = P$ is determined by the pdf f.

* 23) Gives a method to construct new measures ν from μ.   do Qual problems

§124 Product measures, Fubini Th, Tonelli th.

1) Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two complete measure spaces. Let the cartesian product = direct product = cross product

$X \times Y = \{(x,y) : x \in X, y \in Y\}$.

If $A \subseteq X$ and $B \subseteq Y$, we call $A \times B$ a rectangle.

If $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$,
$A \times B$ is a measurable $\underbrace{\text{rectangle}}_{\text{cartesian product is better}}$

If $A \times B$ is a measurable rectangle,
then the <u>product measure</u> $\lambda = \mu \times \nu$

$$\lambda(A \times B) = \underbrace{\mu(A)\ \nu(B)}_{\text{multiply}}.$$

2) Let $\mathcal{A}$ = collection of all measurable rectangles: $\mathcal{A} = \{ A \times B : A \in \mathcal{F}_1, B \in \mathcal{F}_2 \}$.
"analog" of open sets, closed sets etc

3) Let $\mathcal{F} = \sigma(\mathcal{A})$ be the smallest σ-algebra containing $\mathcal{A}$.

Then $(X \times Y, \mathcal{F}, \lambda)$ is a measure space.

ex) If $X = Y = \mathbb{R}$ and $\mu = \nu$ =
two-dimensional L. measure for $\mathbb{R}^2$ = plane.

501 Qual problems

1} Try to memorize Definitions, Theorems and proofs from Royden, especially proofs from Ch1-4. Buy or borrow The Memory Book.

2} Some universities include problems from Math 352. SIU often includes problems from functional analysis. See link to old quals.

3} Try to get and work Royden qual problems. Schaum's outline from Syllabus

ex} Let f be a measurable function in a set E. Suppose G is an open set and F is a closed set. Are

i) $\{x \in E : f(x) \in G\}$ and ii) $\{x \in E : f(x) \in F\}$ measurable? Justify your answers

Schaum outline 3.15!

Proof i) Let $G = \bigcup_{k=1}^{\infty} I_k$ where

$I_k = (a_k, b_k)$ are the disjoint component intervals. Let $f: E \to \mathbb{R}$. Then

Yes, i) $= \{x \in E : f(x) \in \bigcup_{k=1}^{\infty} (a_k, b_k)\}$

$$= \bigcup_{k=1}^{\infty} \{x \in E : f(x) \in (a_k, b_k)\}$$

f is a function and the I_k are disjoint. $\therefore$, $f(x) \in G$ iff $f(x) \in$ exactly one I_k, say I_{k_0}.

$$= \bigcup_{k=1}^{\infty} \left[\underbrace{\{x \in E : f(x) > a_k\}}_{\in \mathfrak{M} = \mathcal{F}_M} \cap \underbrace{\{x \in E : f(x) < b_k\}}_{\in \mathfrak{M} = \mathcal{F}_M} \right]$$

$\in \mathcal{F}_M$ since $\mathcal{F}_M$ is a σ-algebra and f is measurable.

Note that i) $= f^{-1}(G)$.

$\not{\text{les}}$ (← around notes 42)

Sol @ 2)

yes, ii) $= f^{-1}(F)$ $\underset{\text{say}}{=}$ ... $F = G_F^c$ where G_F is open

$$\{x \in E : f(x) \in G_F^c\} = E - \{x \in E : f(x) \in G_F\}$$

$$= \underbrace{E}_{\in \bar{\partial} m} \cap \underbrace{\{x \in E : f(x) \in G_F\}^c}_{\in \bar{\partial} m \text{ by i)}}$$

$\therefore$ ii) $\in \bar{\partial} m = \mathfrak{M}$

$\square$

ex} Let f be L_1 integrable on $[a,b]$.

Let $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$.

Show that F is of BV.

Soln} F is absolutely continuous.

$\therefore$ F is of BV. $\square$

Alternatively use the proof of Lemma 5.7 on p105: Let $a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a,b]$.

Then $t = \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| =$

$$\sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\,dt$$

$$= \int_a^b |f(t)|\,dt < \infty$$

Hence $T = \sup_{\pi} t \leq \int_a^b |f(t)|\,dt < \infty$

$\therefore$ f is of BV.

$\square$

ex3 Let f and f_n be measurable and nonnegative functions on $[0,1]$. Suppose

$\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in $[0,1]$ and

$\lim_{n\to\infty} \int_{[0,1]} f_n\,dx = \int_{[0,1]} f\,dx < \infty$.

Show that if E is any measurable set in $[0,1]$, then $\int_E f_n \to \int_E f$.

proof) (The generalization to the LDCT is Th 4.17: Let g_n be a sequence of integrable functions which converges ae to an integrable function g. Let f_n be a sequence of measurable functions such that $|f_n| \le g_n$ and $f_n \to f$ ae. If $\int g = \lim_{n\to\infty} \int g_n$, then $\int f = \lim_{n\to\infty} \int f_n$.) 501 Q 3)

Now $|f_n \chi_E| = \underbrace{f_n \chi_E}_{h_n} \le \underbrace{f_n \chi_{[0,1]}}_{g_n}$

since f_n is nonnegative.

$$\left.\begin{aligned} \underbrace{\int_{[0,1]} f}_{\int g} &= \lim_{n\to\infty} \int_{[0,1]} f_n \\ &= \lim_{n\to\infty} \int g_n \end{aligned}\right\} \text{given}$$

By Th 4.17, $\int h = \lim_{n\to\infty} \int h_n$ where $h_n = f_n\chi_E \to f\chi_E = h$ ae

or $\int f\chi_E = \lim_{n\to\infty} \int f_n \chi_E$

or $\int_E f_n \to \int_E f$.

$\square$

Sept 1998 Qual 501 SIU

Give an example of a measurable space $(X, \mathcal{F})$ and a function $f: X \to \mathbb{R}$ which is not $\mathcal{F}$ measurable, but is such that $|f|$ and f^2 are $\mathcal{F}$ measurable.

Soln: Let $\mathcal{F} = \mathfrak{M} = \mathcal{F}_m$ for L measurable functions. Let $X = \mathbb{R}$. Let $V \subseteq \mathbb{R}$ be a non measurable function.

$$\text{Let } f(x) = \begin{cases} 1 & x \in V \\ -1 & x \notin V \end{cases}$$

Then f is not L measurable (not measurable $\mathcal{F}_m$), but $|f| = f^2 = g(x) \equiv 1 \ \forall x \in \mathbb{R}$ is L measurable.

See HW 9 Problem 3

Counts ch1–4

501 Final or qual
2010 Xiao

1. Let f be a real-valued function with domain D
(2pts) (i) Give the definition of f being continuous on D;
(7pts) (ii) Let f be

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \quad \text{on } \mathbb{Q}^c \cap D \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest terms,} \quad \text{on } \mathbb{Q} \cap D \end{cases}$$

where p, q are integer and $q > 0$. At what points is f continuous? You need to prove your claim.

$|f(x)| \leq 1 = \frac{1}{1}$

p

$(-2,1)$ $(-2,2)$ $(-2,3)$ …
$(-1,1)$ $(-1,2)$ $(-1,3)$ …
$(0,1)$ $(0,2)$ $(0,3)$ …
$(1,1)$ $(1,2)$ $(1,3)$ …
$(2,1)$ $(2,2)$ $(2,3)$ …
$(3,1)$ $(3,2)$ $(3,3)$ …

x

Let $\varepsilon > 0$, $x_0 \in D$.

Let $x \in (x_0 - \delta, x_0 + \delta) \cap D$

ii) Then $|f(x) - f(x_0)| \leq \frac{1}{q} < \varepsilon$ if $q > \frac{1}{\varepsilon}$

$|f(x) - f(x_0)| > \varepsilon$ if $\frac{1}{q} > \varepsilon$ or $q < \frac{1}{\varepsilon}$

If $x_0 \in D$ is not an accumulation point of D then $f(x)$ is continuous at x_0.
If D is a nonempty interval, then for any $x_0 \in D$ and any $\varepsilon > 0$ if $\delta > 0$ then the rationals are dense so there will be an $x \in D \cap (x_0 - \delta, x_0 + \delta)$ such that $q_x < \frac{1}{\varepsilon}$ and $|f(x) - f(x_0)| > \varepsilon$.
Then f is nowhere continuous on D.
(can handle some other domains D in a similar manner.)

not 6.22 not Problem 6.24 (not Th 5.11 p 74) } Schaum

6.(10pts) Let $E \subset R$ be a measurable set. Suppose that $f(x)$ is (Lebesgue) integrable on E. Show that if for any bounded measurable function φ, $\int_E f(x)\varphi(x)dm = 0$, then $f(x) = 0$ a.e. on E.

(*) If $g \geq 0$ and $\int_E g = 0$ then $g = 0$ ae on E.

Let $E^+ = \{x \in E : f(x) \geq 0\}$ and

$E^- = \{x \in E : f(x) \leq 0\}$. Then

$E^+, E^- \in \mathcal{F}_m$ since f is measurable.

Then $\int_E f \chi_{E^+} = \int_E f^+ = 0 \Rightarrow f^+ = 0$ ae on E,

bounded, measurable

$(*)$

and $\int_E f \chi_{E^-} = -\int_E f^- = 0 \Rightarrow f^- = 0$ ae on E.

Hence $f = f^+ - f^- = 0$ ae on E.

$\square$

$\varphi = \chi_A$ is bounded and measurable if $A \in \mathcal{F}_m$ is a measurable set.

(Note. $f^+ = f\chi_{E^+} = \max(f, 0)$ on E
$f^- = -f\chi_{E^-} = -\min(f, 0)$ on E.)