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5} In B, μ and ν are complete measures. Then $\lambda = \mu \times \nu$ is a complete measure.

6} Sometimes write $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 = \sigma(A)$.

7} Know Fubini's Theorem: Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two complete measure spaces and f an integrable function on $X \times Y$. Then

i) For almost all x (ae), the function $f_x(y) = f(x,y)$ is an integrable function on Y.

ii) For almost all y, the function $f^y(x) = f(x,y)$ is an integrable function on X.

iii) $\int_Y f(x,y)\, d\nu(y)$ is an integrable function on X.

iv) $\int_X f(x,y)\, d\mu(x)$ is an integrable function on Y.

v) $\int_X \left[\int_Y f\, d\nu\right] d\mu = \int_Y \left[\int_X f\, d\mu\right] d\nu$

$= \int_{X \times Y} f \; d(\mu \times \nu)$.

8) For Lebesgue integrals, Fubini's theorem is used to prove

$$\iint_{\Omega} f(x,y)\, dx\, dy = \int_{X_1} \left[\int_{Y_1} f(x,y)\, dy\right] dx$$

$$= \int_{Y_2} \left[\int_{X_2} f(x,y)\, dx\right] dy .$$

So double integrals can be computed by iterated integrals.

Here Ω is the region of integration, X_i and Y_i the limits of integration for the single integrals. eg Ω the unit square

$\int_{X_1}, \int_{X_2}, \int_{Y_1}, \int_{Y_2} = \int_0^1$.

9] Know Tonelli's Theorem.

Let $(X, \mathcal{F}_1, \mu)$ and $(Y, \mathcal{F}_2, \nu)$ be two σ-finite measure spaces. Let f be a nonnegative measurable function on $X \times Y$. Then

i) for almost all x, the function $f_x(y) = f(x,y)$ is a measurable function on Y.

ii) For almost all y, the function $f^y(x) = f(x,y)$ is a measurable function on X.

iii) $\int_Y f(x,y)\, d\nu(y)$ is a measurable function on X.

iv) $\int_X f(x,y)\, d\mu(x)$ is a measurable function on Y.

v) $\int_X \left[\int_Y f\, d\nu \right] d\mu = \int_Y \left[\int_X f\, d\mu \right] d\nu$

$= \int_{X \times Y} f\, d(\mu \times \nu)$.

Other Material

§3.6

1) Ch 3 · p 74 **Lusin's Th:** Let f be a measurable real valued function on $[a,b]$. Given $\delta > 0$, there is a continuous function ϕ on $[a,b]$ such that $m(\{x : f(x) \neq \phi(x)\}) < \delta$.

2) This theorem goes with Littlewood's principle that every L. measurable function is nearly continuous.

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1) Lemma 5.11. If f is absolutely continuous on $[a,b]$, then f is of bounded variation on $[a,b]$.

Proof: Let δ be the δ in the def of absolute continuity that corresponds to $\epsilon = 1$. Then any subdivision of $[a,b]$ can be split (by inserting fresh division points if necessary) into

k sets of intervals, each of total length $< \delta$, where

$k = \lfloor 1 + (b-a)/\delta \rfloor$.

$0 = x_0 = a \leq \underbrace{x_1 < x_2}_{1 \quad 2} < \cdots < \underbrace{x_{k-1} < x_k}_{k} = b$

interval

Since $|x_i - x_{i-1}| < \delta$, $|f(x_i) - f(x_{i-1})| < \varepsilon$

(using $n=1$ in the def of absolutely continuous).

Thus $t = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{k} 1 = k$.

$\therefore T = \sup_{\Pi} t \leq k$ and f is of BV.

$\square$

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2) Prove that if f is absolutely continuous on $[a,b]$, then f is continuous on $[a,b]$

proof) If $f(x)$ is absolutely continuous, then given $\varepsilon > 0$, we can find $\delta > 0$ $\ni$

$\sum_{i=1}^{n} |f(x_i + h_i) - f(x_i)| < \varepsilon$ whenever $\sum_{i=1}^{n} \overbrace{|h_i|}^{=h_i} < \delta$

Take $n = 1$ to get $|f(x+h) - f(x)| < \varepsilon$

whenever $|h| < \delta$. $\therefore$ f is continuous

$\square$

§5.1 1] Def. Let Λ be a collection of intervals. Then Λ covers a set E in the sense of Vitali, if for each $\varepsilon > 0$ and any $x \in E$, $\exists$ $I \in \Lambda$ $\ni$ $x \in I$ and $l(I) < \varepsilon$. Λ is a Vitali covering

2] Vitali's covering Lemma. (Lemma 5.1): Let E be a set of finite outer measure and Λ a collection of intervals that cover E in the sense of Vitali. Then given $\varepsilon > 0$, $\exists$ a finite disjoint collection $\{I_1, \ldots, I_N\}$ of intervals in Λ such that $m^*\left(E - \bigcup_{i=1}^{N} I_i\right) < \varepsilon$.

§4.5 convergence in measure

1) Suppose $f_n : D \to \mathbb{R}$, $f : D \to \mathbb{R}$ and $f_n \to f$ a.e. on D. Then

$$m(\{x \in D : f_n(x) \not\to f(x)\}) = 0$$

and $m(\{x \in D : f_n(x) \to f(x)\}) = m(D)$.

So $m(\{x \in D : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$.

2) In 1) can have $f_n \to f$ a.e. μ in D and change m to μ.

3) Let $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$. The sequence of measurable functions f_n converge to f in measure

if, given $\varepsilon > 0$, $\exists N \ni$ for all $n \geq N$, $m(\{x \in D : |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon$.

So $\forall \varepsilon > 0$, $\lim_{n\to\infty} m(\{x \in D : |f(x) - f_n(x)| \geq \varepsilon\}) = 0$

4) Can replace m by μ.

often use δ here

If $\mu = P$ is a probability measure,
then convergence in measure is
convergence in probability
(consistent estimator M580, 581 or 582)

5) If $f_n \to f$ ae in D
where f_n, f go with $(X, \mathcal{F}, \mu)$,
then $f_n \to f$ in measure-μ,

SIU F04 #5 F00 #4 501 Q probs 4)

ex} If $J \subseteq [0,1]$ is an interval, let $L(J)$ denote the length of J. For $A \subseteq [0,1]$, define $\lambda^*(A) = \inf \sum_{K=1}^{N} L(J_K)$ where the inf is taken over all N and all finite families of intervals that satisfy $A \subseteq \bigcup_{K=1}^{N} J_K$.

a) Prove λ^* is finitely subadditive.

b) Prove that λ^* is not countably subadditive.

soln} m^* on $[0,1]$ uses all countable collections of open intervals that cover A.

b) Let $A = \mathbb{Q} \cap [0,1]$. Then A is dense in $[0,1]$, so if we cover A with a finite collection of intervals, then the union of the intervals needs to contain $[0,1]$. Hence the lengths sum to at least one. Thus $\lambda^*(A) = 1$.

use $[0,1]$ as the cover

If $x_i \in A$ then $\lambda^*(\{x_i\}) = 0$

since $(x_i - \delta, x_i + \delta)$ covers $\{x_i\}$ with length

$2\delta > 0$ arbitrary.

$$\therefore\ 0 = \underbrace{\sum_{x_i \in A}}_{\text{countable}} \lambda^*(\{x_i\}) < \lambda^*\Big(\underbrace{\bigcup_{x_i \in A}}_{\text{countable}} \{x_i\}\Big) = \lambda^*(A)$$

a) Show $\lambda^*(A_1 \cup A_2) \le \lambda^*(A_1) + \lambda^*(A_2)$.

Then finite subadditivity follows by induction

Let Λ_i be a finite cover of A_i such

that $\sum_{i \in \Lambda_1} L(I_i) \le \lambda^*(A_1) + \frac{\varepsilon}{2}$ and

$$\sum_{j \in \Lambda_2} L(I_j) \le \lambda^*(A_2) + \frac{\varepsilon}{2}.$$

$\Lambda_1 \cup \Lambda_2$ is a finite cover of $A_1 \cup A_2$

Then $\lambda^*(A_1 \cup A_2) \le \sum_i L(I_i) + \sum_j L(I_j)$

$\le \lambda^*(A_1) + \lambda^*(A_2) + \varepsilon$ where $\varepsilon > 0$

is arbitrary. $\square$ similar to proof of prop 3.2 on p. 57

NIU 93

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ex) prove $A = \int_0^\pi x^{-1/4} \sin x \, dx \le \pi^{3/4}$

Soln $A \le \sqrt{\int_0^\pi x^{-1/2} dx} \sqrt{\int_0^\pi \sin^2 x \, dx}$

$\uparrow$ Cauchy Schwarz $\quad (x^{-1/2})^2 = x^{-1/4}$

$\sqrt{2}\,\pi^{3/4}$ would be better

and $\int_0^\pi x^{-1/2} dx = \dfrac{x^{1/2}}{1/2} \Big|_0^\pi = 2\sqrt{\pi}$

$$\left(\text{while } \int_0^\pi [\sin x]^2 dx \le \int_0^\pi 1 \, dx = x\Big|_0^\pi = \pi \right.$$

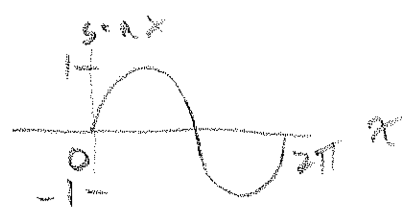
$$\left. \therefore A \le \sqrt{2}\, \pi^{1/4} \sqrt{\pi} = \sqrt{2}\, \pi^{3/4} \right)$$

anti-derivative hard to remember

$$\int_0^\pi \sin^2 x \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} \Big|_0^\pi$$

$$= \left[\frac{\pi}{2} - 0\right] - \left[0 - 0\right] = \frac{\pi}{2}$$

$$\therefore A \le \sqrt{2}\, \pi^{1/4} \sqrt{\frac{\pi}{2}} = \pi^{1/4}\sqrt{\pi} = \pi^{3/4}$$



ex} NIU 93 Final 19), T or F justify

Every absolutely continuous function on $[a,b]$ is differentiable ae on $[a,b]$.

soln T, f is AC $\Rightarrow$ f is BV $\Rightarrow$

$f = g - h$ where $g\uparrow$ and $h\uparrow$ $\Rightarrow$ f is differentiable ae since monotone functions are differentiable a e.

ex} NIU 93 Final

Let $f = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1] \\ \ln(1+x) & x \in [0,1] \setminus \mathbb{Q} \end{cases}$

Find (R) $\int_0^1 f(x)\,dx$ and $\int_{[0,1]} f(x)\,dx$ if they exist.

soln} f is not Riemann integrable since

$m(\{x \in [0,1] : f \text{ is not continuous at } x\}) = 1$

$f = \ln(1+x)$ a e on $[0,1]$.

$\therefore \int_{[0,1]} f(x)\,dx = \int_0^1 \ln(1+x)\,dx = \int_1^2 \ln(u)\,du$

$u = 1+x \quad du = dx$

$x=0 \to u=1 \quad x=1 \to u=2$

$= [u\ln(u) - u]\Big|_1^2 = 2\ln(2) - 2 - [0 - 1] = 2\ln 2 - 1$

hard to remember

ex) NIU June 1993

Let E be a L measurable subset of $[0,1]$.

a) Show that if $m(E)=1$, then E must be dense in $[0,1]$.

b) Show that if $m(E)=0$, then E must have empty interior.

c) Is the converse to either (a) or (b) true? Explain

soln) a) Suppose E is not dense in $[0,1]$.

Then the closure $\bar{E} \neq [0,1]$. So

$\exists\ y \in [0,1]-\bar{E} \ni \exists\ \delta > 0 \ni$

$$\bar{E} \cap \left[[0,1] \cap (y-\delta, y+\delta) \right] = \phi.$$

$$\text{so } m\Big(\overbrace{\bar{E} \cap [0,1]}^{E} \cap (y-\delta, y+\delta)\Big) = 0$$

$$\text{but } m\left[\bar{E} \cap [0,1] \cap (y-\delta, y+\delta) \right] \underbrace{> \delta}$$

$\Rightarrow \Leftarrow$

Choose $\delta > 0$ $\ni y-\delta \in [0,1]$ or $y+\delta \in [0,1]$

(Let $D = [0,1] - E$ so $\overset{\text{disjoint}}{D \cup E} = [0,1]$.

then $m([0,1] \cap (y-\delta, y+\delta)) = m((D \cup E) \cap (y-\delta, y+\delta)) =$

$$\underbrace{m(D \cap (y-\delta, y+\delta))}_{0} + m(E \cap (y-\delta, y+\delta)) = m(E \cap (y-\delta, y+\delta))$$

Sol Final 509

ex) Let λ^* be L outer measure on $\mathbb{R}$,

Prove the following

a) If $A \subseteq \mathbb{R}$ is bounded then $\lambda^*(A) < \infty$. Is the converse of this statement true? If yes prove it, if no give a counterexample.

b) If $E \subseteq [0,1]$ is a L measurable set such that $\lambda(E) = 1$, prove E is dense in $[0,1]$.

c)

soln b) done in last ex

a) Suppose $A \subseteq [-t, t]$.

Then $\lambda^*(A) \leq \lambda^*([-t,t]) = 2t < \infty$.

Let $A = \mathbb{N} = \{1, 2, \ldots\}$.

Then $\lambda^*(A) = 0 < \infty$, but A is unbounded. So the converse is false.