When the Riemann integral exists it is almost always equal to the Lebesgue integral.

An exception is a function with infinite area above and below the x axis (so $\int |f| = \infty$) where the areas cancel so the Riemann integral exists. $f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is an example, and its Riemann integral $\int_0^\infty f(x) = \frac{\pi}{2}$, $\int_{-\infty}^{\infty} f(x) = \pi$ is rather hard to integrate.

Th} (4.0) If $|f|$ is Riemann integrable on E ($E \in \mathfrak{M}$), then f is L. integrable on E, and $\int_E f = (R)\int_E f$; the Lebesgue and Riemann integrals agree.

Keeping the last Th in mind, typically

a) f is Riemann integrable on E

and $\int_E f = (R)\int_E f$

b) $f = g$ a.e in E where g is Riemann integrable on E and $\int_E f = (R)\int_E g$.

c) $f = \chi_A$ and $\int_E \chi_A = \int \chi_{A \cap E} = m(A \cap E)$

where often $E = \mathbb{R}$ and $A \cap E = A$

d) $\int |f| = \infty$

e) $\int \chi_V$ does not exist since

V is not a measurable set and χ_V is not a measurable function

ex) $E = [0,1]$, $f_n(x) = \begin{cases} n & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} \le x \le 1 \end{cases} = n\chi_{[0,\frac{1}{n}]}$

$\int_E f_n = n\int \chi_{[0,\frac{1}{n}]} = n\frac{1}{n} = 1 \quad \forall n$

$= \int_0^{\frac{1}{n}} n\,dx = nx\Big|_0^{1/n} = n\frac{1}{n} = 1 \quad \forall n$

$\therefore \lim\limits_{n\to\infty} \int_E f_n = \lim\limits_{n\to\infty} 1 = 1$

$f_n \to f = \begin{cases} \infty & x = 0 \\ 0 & 0 < x < 1 \end{cases}$, $\int_E f = 1 \cdot 0 = 0$

LDCT not violated because $|f_n| \le g$, g integrable is not satisfied

509 Final 2) week 15 notes linearity for bounded measurable functions 501 Goals 2)

5) State Fatou's Lemma, MCT, use Fatou's Lemma to prove MCT

500 Final 1) Distinguish between an algebra and a σ algebra

2) Define what is meant by L. outer measure λ^* on $\mathbb{R}$.

3) " " L measurable set on $\mathbb{R}$

" " Borel measurable set in $\mathbb{R}$

ex) If $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then the translation $x+A = \{x+a : a \in A\}$.

If $x>0$, $x+A$ is obtained by moving the set to the right x units on the real line.
If $x<0$, $x+A$ " " left $|x|$ "

Prove $m^*(x+A) = m^*(A)$ (can use m if $A \in \mathfrak{M}$): Let $I_1, I_2 \ldots$ be a sequence of open intervals whose union contains A. Then $t+I_1, t+I_2, \ldots$ is a sequence of open intervals whose union contains $t+A$.

$$\therefore \; m^*(t+A) \le \sum_{k=1}^{\infty} \ell(t+I_k) = \sum_{k=1}^{\infty} \ell(I_k). \quad (*)$$

Taking inf of the last term over all sequences of open intervals whose union contains A gives $m^*(t+A) \le m^*(A)$.

To get the other direction, note that $A = -t + (t+A)$.

$$\therefore \; m^*(A) = m^*(-t + (t+A)) \le m^*(t+A)$$

using $(*)$ with A replaced by $t+A$ and t replaced by $-t$.

$$\left(m^*(A) = m^*(-t + (t+A)) \le \sum_{k=1}^{\infty} \ell(I_k) = \underbrace{\sum_{k=1}^{\infty} \ell(t+I_k)}_{\substack{\text{inf of this term} \\ = m^*(t+A)}} \right)$$

$\square$

Similar to 500 2a final

See old exam 2 # 3

Qual 8)

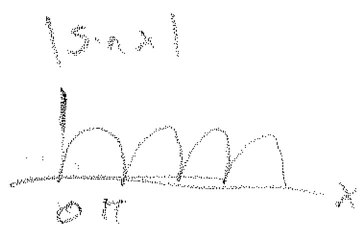
ex) Consider the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=\frac{\sin x}{x}$ $\forall x>0$.

Show the improper Riemann integral $(R)\int_0^\infty f(x)$ exists but the L integral $\int_0^\infty f(x)\,d\lambda(x)$ does not exist.

Hint: prove the series $\sum_{n=0}^{\infty}\int_{n\pi}^{(n+1)\pi}\frac{\sin x}{x}\,dx$ is conditionally convergent

Soln) $\int_1^{n\pi}\left|\frac{\sin x}{x}\right|dx=\int_1^{n\pi}\frac{|\sin x|}{x}\,dx$

$\geq \sum_{k=2}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{x}\,dx$



$\geq \sum_{k=2}^{n}\int_{(k-1)\pi}^{k\pi}|\sin x|\,dx$

$=\sum_{k=2}^{n}\frac{2}{k\pi}=\frac{2}{\pi}\sum_{k=2}^{n}\frac{1}{k}$

$\int_0^{\pi}\sin x\,dx=-\cos x\Big|_0^{\pi}=-\cos(\pi)+\cos(0)$
$=1+1=2$

Now $\sum_{k=2}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$

(calc 1 or $\sum_{k=2}^{n} \frac{1}{k} \geq \sum_{k=2}^{n} \int_{k}^{k+1} \frac{1}{x}\, dx$

$= \int_{2}^{n+1} \frac{1}{x}\, dx = \log(n+1) - \log(2) \to \infty$ as $n \to \infty$),

$\therefore \lim_{n\to\infty} \int_{1}^{n\pi} \left|\frac{\sin x}{x}\right| dx = \int_{1}^{\infty} \left|\frac{\sin x}{x}\right| dx$

$\underline{= \infty}$ and $\int_{0}^{\infty} \left|\frac{\sin x}{x}\right| dx \geq \int_{1}^{\infty} \left|\frac{\sin x}{x}\right| dx = \infty$,

Now $\int_{1}^{n\pi} \frac{\sin x}{x}\, dx = \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{\sin x}{x}\, dx$

$= \sum_{k=2}^{n} a_k$ where the a_k alternate in

sign, $|a_k| \downarrow$ and $a_k \to 0$ as $k \to \infty$.

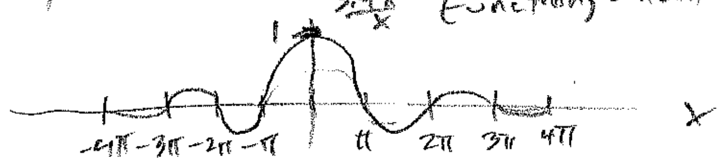
$\therefore \lim_{n\to\infty} \int_{1}^{n\pi} \frac{\sin x}{x}\, dx = \int_{1}^{\infty} \frac{\sin x}{x}\, dx = \sum_{k=2}^{\infty} a_k < \infty$

by the alternating series test.

$\square$

See 909 Sol final 3), Sol Qual 2011 #6

$\frac{\sin x}{x}$ Function, $\text{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$



1) <u>σ-algebra problems</u>

a) Λ nonempty index set, $\mathcal{F}_\lambda$ a σ-algebra on X prove $\mathcal{F} = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is a σ-algebra

old Q2 #3, Hw2 #1 old exam 1 #1

b) Show $\sigma(\mathcal{A})$ is a σ algebra

Hw 2 #2

c) problems like $\sigma(\mathcal{E}) = \mathcal{B}(\mathbb{R})$

Hw4 1), 3), 4)

old Q4 #1 old exam 1 #5

2) Show that certain functions are <u>measurable</u> $\in \mathcal{L}(D)$.

old Exam 2 # 2, old Q6 #2, old Q7 #3

3) Prove $f: \mathbb{R} \to Y$ <u>is measurable</u> if eg $\{x: f(x) \geq t\} \in \mathfrak{M}$ $\forall t \in \mathbb{R}$.

old Exam 2 #4 old Q6 #3

old Q7 #4

4) old exam 3 #1

5) old exam 3 #4 1st half or 2nd half of LDCT proof

6) Integration

a) Prove things like i) MCT

ii) old exam 3 #6

iii) f is L integrable iff $|f|$ is

old exam 3 #5

iv) $|\int_E f| \leq \int_E |f|$

v) $\int_E c = c\, m(E)$

vi) If $m(E) = 0$, $\int_E f = 0$

vii) If $E = \bigcup_{i=1}^{n} E_i$ where the E_i are disjoint

then $\int_E f = \sum_{i=1}^{n} \int_{E_i} f$

viii) linearity

ix) monotonicity

7) Be able to state theorems and Definitions from the exam reviews

8) If $m(A) > 0$, then A is not a countable set.

eg $[0,1] = (\mathbb{Q} \cap [0,1]) \cup (\mathbb{Q}^c \cap [0,1])$

$m([0,1]) = 1 = 0 + \underbrace{m(\mathbb{Q}^c \cap [0,1])}_{1}$

So $\mathbb{Q}^c \cap [0,1]$ is not countable, and $[0,1]$ is not countable (diagonal method often works, but is harder). See old Q4 #3

9) m^* proofs: i) $m^*(\phi) = 0$

ii) monotonicity

iii) finite subadditivity

iv) E countable $\Rightarrow m^*(E) = 0$

v) m^* is not finitely additive:

if V is not measurable then $\exists A \subseteq X$

$\ni m^*(A) \neq m^*(A \cap V) + m^*(A \cap V^c)$

10) $\mathfrak{M} = \mathcal{F}_m$ = σ-algebra of measurable sets

i. $X = \mathbb{R}$, ϕ, $\in \mathcal{F}_m$ $\quad E \in \mathcal{F}_m$ iff $E^c \in \mathcal{F}_m$

11) If $f : [a,b] \to \mathbb{R}$ is of BV

i) $f = g - h$ where g and h are nondecreasing functions

ii) f' exists ae on $[a,b]$

12) If $f : [a,b] \to \mathbb{R}$ is AC then

i) f is of BV

ii) f' exists ae on $[a,b]$

iii) f is continuous on $[a,b]$ proof

13) If f is a L. integrable function on $[a,b]$ then its indefinite integral is

$F(x) = \int_a^x f(t)\,dt$.

F is AC so F is of BV

so F' exists ae on $[a,b]$.

14) measures and disjoint sets

$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, $A_1, \ldots$ disjoint