9/3/4

last time

Let $X \neq \emptyset$. A nonempty collection $\mathcal{F}$ of subsets of X is a σ-algebra on X

if i) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

and ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

Know!

To prove that $\mathcal{F}$ is a σ-algebra

show 0) $\mathcal{F}$ is nonempty, often

by showing $X \in \mathcal{F}$, and

show i) and ii).

So 4 2 A3 is what is used for proofs.

46} P.19 Know Def. Let $\mathcal{A}$ be a class of subsets of $X \neq \emptyset$. The <u>σ-algebra generated by $\mathcal{A}$</u>, denoted by $\sigma(\mathcal{A})$, is the intersection of all σ-algebras containing $\mathcal{A}$

(elements of $\mathcal{A}$ are subsets of X, so elements of $\mathcal{A}$ are not elements of X)

Warning: to prove $\mathcal{F}$ is a σ-algebra need to show 0) $\mathcal{F}$ is nonempty, eg show $X \in \mathcal{F}$.

47} *Let Λ be the indexset of all σ-algebras containing $\mathcal{A}$. Then Λ is nonempty since the σ-algebra of all subsets of X is in Λ. Then

$\sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$. Thus $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$, since if $\mathcal{F}_\lambda$ is a σ-algebra containing $\mathcal{A}$, then

$\sigma(\mathcal{A}) \subseteq \mathcal{F}_\lambda$.

48} (R #14a) p19): HW2 #2 Qual Problem #2 Proof that $\sigma(\mathcal{A})$ is a σ-algebra. Λ is nonempty

0) $\mathcal{A}, X \in \mathcal{F}_\lambda \; \forall \lambda$ $\therefore \mathcal{A}, X \in \sigma(\mathcal{A})$ $\therefore \sigma(\mathcal{A})$ is nonempty

i) If $A_1, A_2, \ldots, \in \sigma(\mathcal{A})$, then

$A_1, A_2, \ldots \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

Hence $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

Thus $\bigcup_{i=1}^{\infty} A_i \in \sigma(\mathcal{A})$.

ii) If $A \in \sigma(\mathcal{A})$, then $A \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

$\therefore A^c \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

Thus $A^c \in \sigma(\mathcal{A}) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$. $\square$

p27 p.52

49) Def. Let $\mathcal{A}$ be the class of all open intervals in $[0,1]$. Then $\sigma(\mathcal{A}) = \mathcal{B}[0,1]$ is the Borel σ-algebra on $[0,1]$.

50) Def. Let $\underline{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$. Let $\mathcal{A}$ be the class of "rectangles" $\{\underline{x} \in \mathbb{R}^k : a_i < x_i \le b_i, i = 1, \ldots, k\}$ where $a_i, b_i \in \mathbb{R}$. Then $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R}^k)$ is the Borel σ-algebra on $\mathbb{R}^k$.

51) $\mathcal{B}[a,b]$ is similar to 49) with $[0,1]$ replaced by $[a,b]$.

52] The Borel σ-algebras on intervals and cross products of intervals are often the σ-algebras on interest: neither too small nor too big.

53] Let $\mathcal{A}_1$ = class of all open intervals in $[0,1]$, Let $\mathcal{A}_2$ = class of all closed intervals in $[0,1]$. Let $\mathcal{A}_3$ = class of all intervals of the form $[a,b)$ in $[0,1]$. Let $\mathcal{A}_4$ = class of all intervals of the form $(a,b]$ in $[0,1]$.

Then $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}_2) = \sigma(\mathcal{A}_3) = \sigma(\mathcal{A}_4)$

$= \mathcal{B}[0,1]$.

54] Notation $\mathcal{F}$ is a σ-algebra unless otherwise stated.

not in test see E1 rev 12)-17)

55] Def. Let A_n be a sequence of sets,

a) $\overline{\lim} A_n = \limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k =$ $\{x : x \in A_n \text{ for infinitely many } n\}$.

b) $\underline{\lim} A_n = \liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k =$

$\{x : x \in A_n$ for all but finitely many $A_n\}$

c) If $\liminf_n A_n = \limsup_n A_n$, then

$\lim_n A_n = A = \liminf_n A_n = \limsup_n A_n$,

written $A_n \to A$.

56] $x \in \limsup_n A_n$ iff for each n, $\exists k \geq n \ni x \in A_k$ iff x is in infinitely many of the A_n. If n has the role of time, A_n occurs infinitely often.

57] $x \in \liminf_n A_n$ iff $\exists n \ni x \in A_k$ $\forall k \geq n$ iff x lies in all but finitely many the A_n. If n has the role of time, the A_n occur "almost always" = for all but finitely many n.

58] $A_n \uparrow A$ means $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \bigcup_{n=1}^{\infty} A_n$.

$A_n \downarrow A$ means $A_1 \supseteq A_2 \supseteq \cdots$ and $A = \bigcap_{n=1}^{\infty} A_n$. (These sets have limits.)

5.9.3 $\bigcap_{k=n}^{\infty} A_k \uparrow \liminf_n A_n$

$\bigcup_{k=n}^{\infty} A_k \downarrow \limsup_n A_n$.

For every m and n,

$\bigcap_{k=m}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k$,

since for $i \geq \max(m,n)$,

$LHS \subseteq A_i \subseteq RHS$.

Taking union over m and intersection over n shows $\liminf_n A_n \subseteq \limsup_n A_n$.

Also, if x lies in all but finitely many of the A_n ($x \in \liminf_n A_n$) the x lies in infinitely many A_n ($x \in \limsup_n A_n$).

ex3 $A_n = \{(-1)^n\}$. Then $\limsup_n A_n = \{-1, 1\}$

Since both numbers occur infinitely often.

$\liminf_n A_n = \phi$ since -1 and 1 are the only possible elements of A_n and neither number occurs for all but finitely many A_n.

σ algebra

60} Th Let A_n be a sequence of $\mathcal{F}$ sets.

a) $\overline{\lim}\, A_n$, $\underline{\lim}\, A_n \in \mathcal{F}$.

b) If $\lim_n A_n$ exists, $\lim_n A_n = A \in \mathcal{F}$.

c) $\liminf_n A_n \subseteq \limsup_n A_n$

d) $(\limsup_n A_n)^c = \liminf_n A_n^c$

e) $(\liminf_n A_n)^c = \limsup_n A_n^c$

Proof a) $C_n = \bigcup_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n.

Hence $\bigcap_{n=1}^{\infty} C_n = \overline{\lim}\, A_n \in \mathcal{F}$.

$B_n = \bigcap_{k=n}^{\infty} A_k \in \mathcal{F}$ for each n.

Hence $\bigcup_{n=1}^{\infty} B_n = \underline{\lim}\, A_n \in \mathcal{F}$.

b) Follows from a).

c) See 59).

d) By De Morgan's laws applied twice,

$$(\limsup_n A_n)^c = \left[\bigcap_{n=1}^{\infty} C_n\right]^c = \bigcup_{n=1}^{\infty} C_n^c$$

$= \liminf_n A_n^c$ where C_n is given in c).

e) By De Morgan's laws applied twice,

$$(\liminf_n A_n)^c = \left(\bigcup_{n=1}^{\infty} B_n\right)^c = \bigcap_{n=1}^{\infty} B_n^c$$

$= \limsup_n A_n^c$ where B_n is given by b).

Remarks a) In $\limsup_n A_n \subseteq A \subseteq \liminf_n A_n$ then $\lim_n A_n = A$ by 60].

b) $B_n = \bigcap_{k=n}^{\infty} A_k \uparrow \underline{\lim}\, A_n$.

Thus $\lim_{n\to\infty} \bigcap_{k=n}^{\infty} A_k = \underline{\lim}\, A_n$.

c) $C_n = \bigcup_{k=n}^{\infty} A_k \downarrow \overline{\lim}\, A_n$.

Thus $\lim_{n\to\infty} \bigcup_{k=n}^{\infty} A_k = \overline{\lim}\, A_n$, and

$$\overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} C_n.$$

61} Do not treat convergence of sets like convergence of functions.

$A_n \to A$ iff $\limsup_n A_n = \liminf_n A_n$

which implies that if $x \in A_n$ for infinitely many n, then $x \in A_n$ for all but finitely many n.

62} Want to show that open, closed and half open (= half closed) intervals can be written as a countable union or countable intersection of intervals of another type. Then the Borel σ algebra $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C})$ where $\mathcal{C}$ is a class of intervals such as the class of all open intervals.

ex} * Prove the following results.

a) $A_1 \subseteq A_2 \subseteq \cdots$ implies $A_n \uparrow A = \bigcup_{n=1}^{\infty} A_n$.

b) $A_1 \supseteq A_2 \supseteq \cdots$ implies $A_n \downarrow A = \bigcap_{n=1}^{\infty} A_n$.

Proof!