a) For each n, $A = \bigcup_{k=n}^{\infty} A_k$.

Thus $\limsup_n A_n = \bigcap_{n=1}^{\infty} A = A$.

For each n, $\bigcap_{k=n}^{\infty} A_k = A_n$.

Thus $\liminf_n A_n = \bigcup_{n=1}^{\infty} A_n = A$.

b) For each n, $\bigcup_{k=n}^{\infty} A_k = A_n$. Thus

$\limsup_n A_n = \bigcap_{n=1}^{\infty} A_n = A$.

For each n, $\bigcap_{k=n}^{\infty} A_k = A$. Thus

$\liminf_n A_n = \bigcup_{n=1}^{\infty} A = A$.

ex3 Let $a < b$

a) $I = \bigcap_{n=1}^{\infty} (a, b+\frac{1}{n}) = (a, b]$.

(common error: say $I = (a, b)$.)

b) $I = \bigcup_{n=1}^{\infty} (a, b-\frac{1}{n}] = (a, b)$

(common error: say $I = (a, b]$.)

c) $I = \bigcup_{n=1}^{\infty} \underbrace{\left[a+\frac{1}{n}, b-\frac{1}{n}\right]}_{A_n \uparrow I} = (a, b)$

d) $I = \bigcap_{n=1}^{\infty} [a, b+\frac{1}{n}) = [a, b]$.

e) $I = \bigcap_{n=1}^{\infty} [a, a+\frac{1}{n}) = [a, a] = \{a\}$.

f) $I = \bigcup_{n=1}^{\infty} [a, b-\frac{1}{n}] = [a, b)$.

Tip! find the interval endpoints eg a and b, and deterine if the interval is open or closed at the endpoint.

warning! students often make errors for this type of problem.

proof of a) $\bigcap_{n=1}^{\infty} (a, b+\frac{1}{n}) = \bigcap_{n=1}^{\infty} A_n$ where

$A_n \downarrow I = (a, b] = A$.

Note that $(a, b] \subseteq A$ since $b \in (a, b+\frac{1}{n})\ \forall n$.

For any $\varepsilon > 0$, $(a, b+\varepsilon] \not\subseteq A$ since

$b+\frac{1}{n} < b+\varepsilon$ for large enough n.

Note that $b+\frac{1}{n} \to b$, but sets are not functions.

Also a and b are the endpoints. $a \notin A_n$ for any n so $a \notin A$ so (open at a. $b \in A_n\ \forall n$ so $b \in A$ so closed at b.

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"Proof of c)" $I = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b-\frac{1}{n}] = \bigcup_{n=1}^{\infty} A_n$

I has endpoints a and b

$a, b \notin A_n = [a+\frac{1}{n}, b-\frac{1}{n}]$ for any n.

$\therefore \{a,b\} \notin A$ $\therefore I = A = (a,b)$.

See 17) in Exam 1 review.

§1.6 63) A sequence $\{x_n\}_{n=1}^{\infty}$ in X

is a function $f: \mathbb{N} \to X$. $(x_1, x_2, \ldots)$

$\therefore f(n) = x_n$, $n \geq 1$.

64) A set is finite if it is $\emptyset$

or the range of a finite sequence.

A set is countable if it is $\emptyset$

or the range of a sequence.

Hence a set X is countable if ($\Leftrightarrow$ iff) $\exists$ an onto

map $f: \mathbb{N} \to X$. So $f[\mathbb{N}] = X$.

X is uncountable if X is not countable.

$\therefore \nexists$ an onto map $f: \mathbb{N} \to X$.

X is countably infinite if X is countable

and infinite.

ex) $\mathbb{N}$ is countable

Let $f_I(x) = x$ for $x \in \mathbb{N}$.

The identity map $f_I : \mathbb{N} \to \mathbb{N}$ is onto.

65) The <u>zigzag map</u> $(A \neq \emptyset, B \neq \emptyset)$

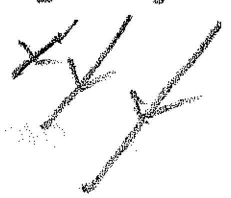
$h_z : \mathbb{N} \to A \times B$

$h_z(1) = (f(1), g(1))$, $h_z(2) = (f(1), g(2))$

$h_z(3) = (f(2), g(1))$ etc

	$g(1)$	$g(2)$	$g(3)$	$\cdots$
$f(1)$	$(f(1), g(1))$	$(f(1), g(2))$	$(f(1), g(3))$	
$f(2)$	$(f(2), g(1))$	$(f(2), g(2))$	$(f(2), g(3))$	$\cdots$
$f(3)$	$(f(3), g(1))$	$(f(3), g(2))$	$(f(3), g(3))$	$\cdots$
$\vdots$				

The diagonal map h_D also works. triangle map



etc.

$\cdots$ $-\frac{4}{1}$	$-\frac{3}{1}$	$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\cdots$
$-\frac{4}{2}$	$-\frac{3}{2}$	$-\frac{2}{2}$	$-\frac{1}{2}$	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\cdots$
$-\frac{4}{3}$	$-\frac{3}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\cdots$

66) Thm $A \times B$ is countable if A is countable and B is countable.

(Since the map in 65) is onto, and $A \times B = \emptyset$ if $A = \emptyset$ or $B = \emptyset$.)

Note! Some elements of the set may be counted more than once. (onto, not always 1-1)

67] Th: A countable union (or collection) of countable sets is countable.

proof: for $B = \bigcup_{i=1}^{\infty} A_i$ where each A_i is countably infinite.

See Royden p22 for general proof

$$\left.\begin{array}{c|cccc} A_1 & a_{11} & a_{12} & a_{13} & \cdots \\ A_2 & a_{21} & a_{22} & a_{23} & \cdots \\ A_3 & a_{31} & a_{32} & a_{33} & \cdots \end{array}\right\}$$

could have duplicates

Then the zigzag map $h_z : \mathbb{N} \to \bigcup_{i=1}^{\infty} A_i$ is onto.

68] $\mathbb{Q}$ = set of rational numbers is countable.

$\mathbb{Q} = \{ \frac{p}{q} : p \in \mathbb{Z}$ and $q \in \mathbb{Z} - \{0\} \}$

Proof: Let $\mathbb{Q}^+ = \{ \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}, p$ and q relatively prime $\}$

$\mathbb{Q}^- = \{ \frac{p}{q} : -p \in \mathbb{N}, q \in \mathbb{N}, p$ and q relatively prime $\}$

Then $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$.

"$-\mathbb{Q}^-$" $= \mathbb{Q}^+$ so $\mathbb{Q}^-$ is countable if $\mathbb{Q}^+$ is.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & \\ \hline 1 & 1/1 & 1/2 & 1/3 & \cdots \\ 2 & 2/1 & 2/2 & 2/3 & \cdots \\ 3 & 3/1 & 3/2 & 3/3 & \cdots \\ \vdots & & & & \end{array}$$

$\mathbb{Q}^+$ is countable

lots of duplicates

eg $1/1$, $2/2$, $3/3$ etc

69] If A_i, $i=1,\dots,n$ is countable,
then $\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \dots \times A_n$
is countable. (By induction, $A_1 \times A_2$ is countable so $(A_1 \times A_2) \times A_3$ is countable etc.)

70] a) Every subset of a finite set is finite.
b) Every subset of a countable set is countable.

proof of b). Let $E = \{x_n\}_{n=1}^{\infty}$ be countable and let $A \subseteq E$.
If $A = \phi$, then A is finite. If $A \neq \phi$, choose $x \in A$. Define new sequence $\{y_n\}_{n=1}^{\infty}$ by setting
$y_n = x_n$ if $x_n \in A$ and $y_n = x$ if $x_n \notin A$.
Then A is the range of $\{y_n\}_{n=1}^{\infty}$ and
$\therefore$ countable. $\square$

§1.5 71] Axiom of Choice: Let $\mathcal{C}$ be any collection of nonempty sets. Then there is a set function F defined on $\mathcal{C}$ which assigns to each set $A \in \mathcal{C}$ an element $F(A)$ in A.

72] "meaning" $F: \mathcal{C} \to \mathbb{R}$ and if

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$\mathcal{C} = \{A_1, A_2, A_3, \dots\}$

then $\exists\ a_i \in A_i \ni F(A_i) = a_i$.

Back to §16

73) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$.

If $\exists$ function $f: \{a_n\}_{n=1}^{\infty} \to \mathbb{R}$ with

$f[\{a_n\}_{n=1}^{\infty}] = E$, then E is countable.

proof: Let function $g(n) = a_n$.

Then $f \circ g: \mathbb{N} \to E$ with $\underbrace{f \circ g[\mathbb{N}] = E}_{\text{onto}}$.

$\therefore E$ is countable.

74) Prop: The set E of infinite sequences from $\{0,1\}$ $(\{x_n\}_{n=1}^{\infty} : x_n \in \{0,1\})$ is not countable.

proof. Assume E is countable.

$\{y_{1k}\}_{k=1}^{\infty} = 1\ 0\ 0\ \cdots\ 0\ \cdots$

$\{y_{2k}\}_{k=1}^{\infty} = 0\ 1\ 0\ \cdots\ 0\ \cdots$

$\{y_{jk}\}_{k=1}^{\infty} = 0\ \cdots\ 0\ 1\ 0\ \cdots\ 0\ \cdots$

jth position

Since $\{y_{jk}\}_{k=1}^{\infty} \in E$ for each $j \in \mathbb{N}$,

E is countably infinite if E is countable.

Let $x_i = \{a_{in}\}_{n=1}^{\infty}$ where $a_{in} \in \{0,1\}$ be the sequences in E.

$$
\begin{array}{c|cccc}
x_1 & a_{11} & a_{12} & a_{13} & \dots \\
x_2 & a_{21} & a_{22} & a_{23} & \dots \\
x_3 & a_{31} & a_{32} & a_{33} & \dots
\end{array}
$$

Now construct sequence $S = \{b_r\}_{r=1}^{\infty}$

where $b_r = 1 - a_{rr}$ for $r = 1, 2, \dots$.

Thus $b_r \in \{0,1\} \Rightarrow \underline{S \in E}$.

But $S = \{b_r\}_{r=1}^{\infty} \neq \{a_{in}\} = x_i$ for any $i \in \mathbb{N}$

Since $b_r = 1 - a_{rr} \neq a_{rr}$.

$\therefore \underline{S \notin E} \Rightarrow\Leftarrow$ (contradiction)

$\therefore E$ is uncountable. $\square$ diagonal technique

75] $\mathbb{R}$ is uncountable.

Proof sketch: Each number x in $[0,1]$ has a binary expansion $0.a_1 a_2 a_3 \dots$

If x is of the form $\frac{1}{2^n}$ take the expansion that ends in all 1's