eg $\frac{1}{2} = 0.100\ldots0\ldots$

$= 0.011\ldots1\ldots = \sum_{i=2}^{\infty} \frac{1}{2^i}$

So $[0,1]$ is uncountable by 74} and $\mathbb{R}$ is uncountable too.

Sh.m of 1.7, 1.8 and 1.9

76} The <u>cartesian product</u>

$$\underbrace{\prod_{\lambda\in\Lambda} X_\lambda}_{\text{(nonempty by the Axiom of choice if each } X_\lambda \neq \emptyset)} = \Big\{ \underbrace{\{x_\lambda\}_{\lambda\in\Lambda}}_{\text{(not a sequence if } \Lambda \text{ is uncountable)}} : x_\lambda \in X_\lambda \ \forall \lambda \in \Lambda \Big\}$$

ex} $\prod_{n\in\mathbb{N}} X_n = \prod_{n=1}^{\infty} X_n =$

$$\Big\{ \underbrace{\{x_n\}_{n=1}^{\infty}}_{\text{sequence}} : x_n \in X_n \ \forall n \geq 1 \Big\}$$

77} $\prod_{i=1}^{\infty} A_i$ tends to be uncountable

Ch. 2 Sol 19

1} $\mathbb{R} = (-\infty, \infty)$ Strong field axioms and order axioms p 31-32.

2} Let $S \subseteq \mathbb{R}$. Then d is an <u>upper bound for S</u> if for each $x \in S$, $x \le d$, and c is a <u>least upper bound for S</u> if c is an upper bound for S and if $c \le d$ for each upper bound of S. (If $c = \sup S \in \mathbb{R}$, i) $\forall x \in S, x \le c$, ii) $\forall \varepsilon > 0, \exists x_0 \in S \ni c - \varepsilon < x_0 \le c$.)

3) d is unique if it exists.

4) p 33 <u>completeness axiom</u>: Every nonempty set S of real numbers which has an upper bound has a least upper bound.

5} Often denote the least upper bound of S by $\sup_{x \in S} x$, $\sup S$.

6} Let $S \subseteq \mathbb{R}$. Then a is a <u>lower bound for S</u> if for each $x \in S$, $x \ge a$.

$b = \inf S = \inf_{x \in S} x$ is a <u>greatest lower bound</u>

if inf S is a lower bound for S

and $\inf S \geq a$ for each lower bound of S. (If $b = \inf S \in \mathbb{R}$
i) $\forall x \in S$, $x \geq b$, ii) $\forall \varepsilon > 0$ $\exists x_0 \in S \ni b \leq x_0 < b + \varepsilon$)

7} $\inf_{x \in S} x = -\sup_{x \in S} -x$

Thus any nonempty set of real numbers with a lower bound has a greatest lower bound.

8} Let $\{x_n\}_{n=1}^{\infty} = \{x_n\}$ be a sequence of real numbers.

$x_n \uparrow x$ if $x_1 \leq x_2 \leq \cdots$ and $x_n \to x$

$x_n \downarrow x$ if $x_1 \geq x_2 \geq \cdots$ and $x_n \to x$.

9} Def. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

a) $\sup a_n$ = least upperbound of $\{a_n\} = \sup_n a_n$

b) $\inf a_n$ = greatest lower bound of $\{a_n\} = \inf_{n \in \mathbb{N}} a_n$

c) $\limsup_n a_n = \overline{\lim}_n a_n$ is the limit of the nonincreasing sequence $\{\sup_{k \geq j} a_k\}_{j=1}^{\infty}$

$= \lim_{n \to \infty} \sup_{k \geq n} a_k = \inf_n \underbrace{\sup_{k \geq n} a_k}$ decreasing seq as $n \uparrow$ so $\sup \downarrow$

d) $\liminf_n a_n = \underline{\lim}_n a_n$ is the limit of the nondecreasing sequence $\{\inf_{k \geq j} a_k\}_{j=1}^{\infty}$.

$= \lim_{n \to \infty} \inf_{k \geq n} a_k = \sup_n \underbrace{\inf_{k \geq n} a_k}_{\text{nondecreasing}}$ set as $n \uparrow \infty$, fewer elements so $\inf \uparrow$

10) Remarks.

a) Unlike the limit, $\overline{\lim}_n a_n$ and $\underline{\lim}_n a_n$ always exist if $\pm\infty$ are allowed as limits, since the limits of nondecreasing and nonincreasing sequences then exist.

b) $\underline{\lim}_n a_n \leq \overline{\lim}_n a_n$

c)* $\lim_{n\to\infty} a_n = a$ iff $\underline{\lim}_n a_n = \overline{\lim}_n a_n = a$.

Hence the limit exists iff $\underline{\lim}_n a_n = \overline{\lim}_n a_n$ (and iff $\overline{\lim}_n a_n \leq \underline{\lim} a_n$ by b)). Again $a = \pm\infty$ is allowed.

d)* Let $\lim^*_n a_n$ be $\overline{\lim}_n a_n$ or $\underline{\lim}_n a_n$.

If $a_n \leq b_n$, then $\lim^*_n a_n \leq \lim^*_n b_n$.

If $a_n < b_n$, then $\lim^*_n a_n \leq \lim^*_n b_n$.

If $a_n \geq b_n$, then $\lim^*_n a_n \geq \lim^*_n b_n$.

If $a_n > b_n$, then $\lim^*_n a_n \geq \lim^*_n b_n$.

Thus, when taking liminf or limsup of both sides of a strict inequality, the $<$ or $>$ must be replaced by $\leq$ or $\geq$.

A similar result holds for limits if both limits exist.

e) $\limsup_n (-a_n) = -\liminf_n a_n$

f) i) $\limsup_n a_n = \overline{\lim}_n a_n$ is the limit of the nonincreasing sequence

$$\sup_{k \geq m} a_k, \sup_{k \geq m+1} a_k, \ldots .$$

ii) $\liminf_n a_n = \underline{\lim}_n a_n$ is the limit of the nondecreasing sequence

$$\inf_{k \geq m} a_k, \inf_{k \geq m+1} a_k, \ldots .$$

iii) $\overline{\lim}_n a_n = \inf_{\substack{n \\ n \in \mathbb{N}}} \overbrace{\sup_{k \geq n} a_k}^{\text{sequence } \{b_j\}_{j=1}^{\infty} = \{\sup_{k \geq j} a_k\}_{k=1}^{\infty}} = \lim_{k \to \infty} \sup(a_n, n \geq k)$

iv) $\underline{\lim}_n a_n = \sup_n \inf_{k \geq n} a_k = \lim_{k \to \infty} \inf(a_n, n \geq k)$.

v)* If a limit point of a sequence $\{a_n\}$ is any number, including $\pm\infty$, that is a limit of some subsequence (of $\{a_n\}$), then $\liminf_n a_n$ and $\limsup_n a_n$ are the inf and sup of the set of limit points, often the smallest and largest limit points.

11) ✓ A limit point is also called 501 21
an accumulation point or a cluster point.

If $\{x_n\}$ is a bounded sequence, then
$\overline{\lim}\, x_n$ = largest limit point of $\{x_n\}$ and
$\underline{\lim}\, x_n$ = smallest limit point of $\{x_n\}$.

12) ✓ * warning! A common error is to take
the limit of both sides of an
equation $a_n = b_n$ or of an inequality,
eg $a_n \le b_n$. Taking the limit is
an error if the existence of the limit
has not been shown. If $\pm\infty$ are allowed
$\underline{\lim}_n\, a_n$ and $\overline{\lim}_n\, a_n$ always exist.
~~Hence~~ the $\underline{\lim}_n\, a_n$ or $\overline{\lim}_n\, a_n$ of the
above equation or inequality can be taken.

ex) a) If $a_n = (-1)^n$ then
$\limsup_n a_n = 1$ and $\liminf_n a_n = -1$
(the 2 limit points).

b) If $a_n = \frac{(-1)^n}{n}$, then

$\limsup_n a_n = \liminf_n a_n = \lim_n a_n = 0$.

c) Note that $\frac{1}{n+1} < \frac{1}{n}$, but

$\lim_n^* \frac{1}{n+1} \leq \lim_n^* \frac{1}{n}$. In fact

$\overline{\lim}_n \frac{1}{n+1} = 0 = \underline{\lim}_n \frac{1}{n+1} = \overline{\lim}_n \frac{1}{n} = \underline{\lim}_n \frac{1}{n}$.

Thus $\overline{\lim}_n \frac{1}{n+1}$ is not less than $\overline{\lim}_n \frac{1}{n}$,

and $\underline{\lim}_n \frac{1}{n+1}$ is not less than $\underline{\lim}_n \frac{1}{n}$.

13) p37 $x_0 \in \mathbb{R}$ is called <u>a cluster point</u> of $\{x_n\}$

if $\forall \varepsilon > 0$ and $\forall N \in \mathbb{N}$, $\exists n \geq N$

$\ni |x_n - x_0| < \varepsilon$. $\therefore$ infinitely many terms of

the sequence $\{x_n\}$ are within ε of x_0.

∞ is a cluster point of $\{x_n\}$ if given $B > 0$ and

given N, $\exists n \geq N \ni x_n > B$

$-\infty$ is a cluster point of $\{x_n\}$ if given $\alpha < 0$

and given N, $\exists n \geq N \ni x_n < \alpha$.

14) $\mathbb{R}_E = \mathbb{R}^* = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ = set of extended real numbers. $\infty - \infty$ is not defined

$0(\infty) = 0 = 0(-\infty)$. $\frac{\infty}{\infty}$ = 0 (real or undefined)?

§2.2 15} Axiom of Archimedes: Given any real number x, there is an integer n $\ni x < n$.

16} Th} If $x < y$ $\exists\, r \in \mathbb{Q}$ (rationals) $\ni x < r < y$.

§2.4 17} Def. $\lim x_n = \lim_{n \to \infty} x_n = L \in \mathbb{R}$

$\forall \varepsilon > 0 \;\; \exists N = N_\varepsilon \ni \forall n \geq N,\; |x_n - L| < \varepsilon.$

18} Def: $\{x_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$, $\exists N = N_\varepsilon \ni \forall n \geq N$ and $\forall m \geq N$, $|x_n - x_m| < \varepsilon$.

19} Th A sequence $\{x_n\}$ converges (in $\mathbb{R}$) iff $\{x_n\}$ is a Cauchy sequence.

§2.5 20} $(a,b]$ and $[a,b)$ are called half open intervals.

21} Notation: let $\mathcal{O}$ and $\mathcal{O}_i$ (oh not zero) denote open sets. $\emptyset$ and $\mathbb{R}$ are open sets.

(there is no $x \in \emptyset$ so the def holds trivially)

22} Def A set $\mathcal{O}$ is an open set of real numbers if $\forall x \in \mathcal{O}$, $\exists\, \delta > 0 \ni (x-\delta, x+\delta) \subset \mathcal{O}$.

Equivalently for each $x \in \mathcal{O}$, $\exists\, \delta > 0 \ni$ each y with $|x - y| < \delta$ belongs to $\mathcal{O}$.

23} Th. The intersection of a finite collection of open sets is open.

proof} (If $\bigcap_{i=1}^{n} \mathcal{O}_i = \phi$, the result holds.)

If $x \in \bigcap_{i=1}^{n} \mathcal{O}_i$ then $\exists\ \delta_i \ni$

$(x-\delta_i, x+\delta_i) \in \mathcal{O}_i$. Take $\delta = \min(\delta_1, \dots, \delta_n)$.

Then $(x-\delta, x+\delta) \in \mathcal{O}_i$ for each i.

$\therefore (x-\delta, x+\delta) \in \bigcap_{i=1}^{n} \mathcal{O}_i$. $\square$

$\mathcal{C} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$

24} Th. The union of any collection of open sets $\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$ is open.

$\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$

proof} Let $x \in \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$. Then $\exists\ \mathcal{O}_{\lambda_0} \ni$

$x \in \mathcal{O}_{\lambda_0}$. Thus $\exists\ \delta = \delta_0 \ni (x-\delta, x+\delta) \in \mathcal{O}_\lambda \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$.

$\square$ (If no such x exists, then $\bigcup_\lambda \mathcal{O}_\lambda = \phi$.)

25} $\bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \bigcap_{i=1}^{\infty} \left(0-\frac{1}{n}, 0+\frac{1}{n}\right) = \{0\} = [0, 0]$

which is not open.

20) Prop 2.8 Let $\mathcal{O} \subseteq \mathbb{R}$ be an open set. Then $\mathcal{O}$ is a countable union of disjoint open intervals.

proof] i) For any $x \in \mathcal{O}$, let

$a_x = \inf \{z : (z, x) \subseteq \mathcal{O}\}$ and let

$b_x = \sup \{z : (x, z) \subseteq \mathcal{O}\}$.

Since $\mathcal{O}$ is open, a_x and b_x are well defined in $\mathbb{R}^*$ (extended reals).

Let $I_x = (a_x, b_x)$.

ii) $b_x \notin \mathcal{O}$ since if $b_x \in \mathcal{O}$, then $\exists\ \delta > 0 \ni (b_x - \delta, b_x + \delta) \in \mathcal{O}$. Thus $(x, b_x + \delta) \in \mathcal{O}$,

ii) contradicting the def of b_x.

iii) Similarly $a_x \notin \mathcal{O}$

iv) $I_x \in \mathcal{O}$ (If $x_0 \in I_x$ and $x_0 \notin \mathcal{O}$, then x_0 contradicts def of a_x or b_x.)

v) Consider the collection of open intervals $\mathcal{C} = \{I_x\}$, $x \in \mathcal{O}$. Then $\mathcal{O} = \bigcup_{x \in \Lambda} I_x$

since $I_x \subseteq \mathcal{O} \Rightarrow \bigcup_{x \in \Lambda} I_x \subseteq \mathcal{O}$ and

if $y \in \mathcal{O}$ then $y \in I_y \Rightarrow \mathcal{O} \subseteq \bigcup_{x \in \Lambda} I_x$,

vi) For any I_x and I_y, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$, proof Suppose $I_x \cap I_y \neq \phi$, $I_x = (a_x, b_x)$, $I_y = (a_y, b_y)$.

Let $w \in I_x$ and $w \in I_y$. Then $a_x = a_y$ and $b_x = b_y$ or there is a contradiction

(If $a_y < a_x$, then $(a_y, b_y) \subseteq \mathcal{O}$ gives a contradiction since a_x is the inf.)

vii) $\{I_x, x \in \mathcal{O}\} = \{I_x, x \in A\} = \mathcal{C}$ where each member is listed once. Hence these I_x are disjoint by vi). (Assume $\mathcal{O} \neq \phi$ since $\mathcal{O} = \phi$ has 1 member countable.)

viii) Let f be a set function $\ni$ $f(I_x) = r_x \in I_x \cap \mathbb{Q}$ (r_x is a rational and $r_x \in I_x$). This f is 1-1, hence A is countable.

$\therefore \mathcal{O} = \bigcup_{I_x \in \mathcal{C}} I_x = \bigcup_{x \in A} I_x$ is a countable union of disjoint open intervals. $\square$

27] Def. A real number x is a <u>point of closure</u> of a set E if $\forall \delta > 0$ $\exists y \in E \ni |x - y| < \delta$.

28] Every $x \in E$ is a point of closure of E.

29] Def] The set of points of closure of E is $\bar{E}$ = closure of E.

30} $x \in \overline{E} \Leftrightarrow \forall \delta > 0,\ (x-\delta, x+\delta) \cap E \neq \emptyset$

$\Leftrightarrow \exists \underbrace{x_n \in E}_{\text{sequence}} \ni x_n \to x.$

31} By 28) $E \subseteq \overline{E}$.

ex} $E = \{\frac{1}{n},\ n \in \mathbb{N}\} \cup \{2\}$.

every subsequence of $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0

$\overline{E} = \{0\} \cup E$

32} $x \notin \overline{E} \Rightarrow \exists\ \delta > 0 \ni (x-\delta, x+\delta) \cap E = \emptyset.$

33) Def} A set F is <u>closed</u> if $F = \overline{F}$.

34) $D^c \subseteq C^c \Rightarrow C \subseteq D$

35) prop 2.10. i) If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (closure not complement)

proof of ii). By i) $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$.

$\therefore \underbrace{\overline{A} \cup \overline{B}}_{D} \subseteq \underbrace{\overline{A \cup B}}_{C}$

Now $x \notin \overline{A} \cup \overline{B} \Rightarrow x \notin \overline{A}$ and $x \notin \overline{B}$.

Thus $\exists\ \delta_1$ and $\delta_2 \ni (x-\delta_1, x+\delta_1) \cap A = \emptyset$

and $(x-\delta_2, x+\delta_2) \cap B = \emptyset$.

Let $\delta = \min(\delta_1, \delta_2)$. $\therefore (x-\delta, x+\delta) \cap (A \cup B) = \emptyset$

$\therefore x \notin \overline{A \cup B}$ by 30}.

So $x \in [\overline{A} \cup \overline{B}]^c \Rightarrow x \in [\overline{A \cup B}]^c$, $\therefore [\overline{A} \cup \overline{B}]^c \subseteq [\overline{A \cup B}]^c$

So by 34), $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$.

$\square$

proof of 34) Suppose $D^c \subseteq C^c$,

Suppose $x_0 \in C$ but $x_0 \notin D$.

Then $x_0 \in D^c \subseteq C^c$ but $x_0 \in C$. $\Rightarrow\Leftarrow$

$\therefore C \subseteq D$. $\square$

36) E is a closed set $\Leftrightarrow \bar{E} \subseteq E$

(Since then $E \subseteq E \Rightarrow E = \bar{E}$.)

37) Prop 2.11. $\bar{E}$ is a closed set.

proof) Need to show $\bar{E} = \bar{\bar{E}}$

$\bar{E} \subseteq \bar{\bar{E}}$ so need to show $\bar{\bar{E}} \subseteq \bar{E}$.

If $x \in \bar{\bar{E}}$, then $\forall \delta > 0$, $\exists y \in \bar{E} \ni$

$|x-y| < \frac{\delta}{2}$. Since $y \in \bar{E}$, $\forall \delta > 0$,

$\exists z \ni |x-z| < \frac{\delta}{2}$.

$\therefore |x-z| \leq |x-y| + |y-z| < \delta \Rightarrow x \in \bar{E}$.

$\therefore \bar{\bar{E}} \subseteq \bar{E}$. $\square$

38) ϕ and $\mathbb{R}$ a closed sets (and open, Some say clopen).

39] prop 2.12
If E_1 and E_2 are closed sets then $E_1 \cup E_2$ is a closed set.
Thus a finite union of closed sets is a closed set.

proof $\overline{E_1 \cup E_2} \underset{\text{by 35]}}{=} \overline{E_1} \cup \overline{E_2} \underset{\text{def of closed set.}}{=} E_1 \cup E_2$

40] prop 2.13 The intersection of any collection $\mathcal{C}$ of closed sets is closed. Let $\mathcal{C} = \{B_\lambda ; \lambda \in \Lambda\}$ be closed sets. Then

$\bigcap_{\lambda \in \Lambda} B_\lambda$ is a closed set.

proof: Let x be a point of closure of $\bigcap_\lambda B_\lambda = E$
Then for any $\delta > 0$, $\exists y \in E \ni |x-y| < \delta$.
Since $y \in B_\lambda$ for each λ, x is a point of closure for each B_λ. $\therefore x \in E$. $\square$

41] prop 2.14 Let $\mathcal{O}$ be an open set and C a closed set. $\mathcal{O}^c$ is closed and C^c is open.

proof] If $x \in \mathcal{O}$ then $\exists \delta > 0 \ni (x-\delta, x+\delta) \in \mathcal{O}$.
$\therefore x$ is not a point of closure of $\mathcal{O}^c$ since there is no $y \in \mathcal{O}^c \ni |x-y| < \delta$. $\therefore \mathcal{O}^c$ contains all of its points of closure and is closed.

If $x \in C^c$, then x is not a point of closure of C. Thus $\exists\, \delta > 0 \ni$ there is no $y \in C$ with $|x-y| < \delta$. $\therefore$ If $|x-y| < \delta$ then $y \in C^c$. $\therefore$ C^c is open. $\square$

p46
42} Def} A set E is called a dense set in B if $\overline{E} = B$

ex) $\mathbb{Q}$ is a dense set in $\mathbb{R}$.

43} Def. A collection $\mathcal{C} = \{E_\lambda : \lambda \in \Lambda\}$ <u>covers</u> F if $F \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$.

$F \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda \Rightarrow \mathcal{C}$ is an <u>open covering</u> of F.

If $\mathcal{C}$ only contains a finite number of sets, then $\mathcal{C}$ is a <u>finite covering</u>.

44) Know <u>Heine Borel Theorem</u>: Let F be a closed and bounded set of real numbers. Then each open covering has a finite subcovering. Ie if $F \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$, then there is a finite collection $\{O_1, \dots, O_n\}$ of