sets in $\Lambda \ni F \subseteq \bigcup_{i=1}^{n} O_i$ 501 26

(where Λ depends on F and $\mathcal{A}$).

proof see text

45) Lindelöf Open Covering Theorem.

Let $\mathcal{C} = \{O_\lambda : \lambda \in \Lambda\}$ be a collection of open sets of real numbers. Then there is a countable subcollection $\{O_i\}$ of $\mathcal{C}$ $\ni$ $\bigcup_{\lambda \in \Lambda} O_\lambda = \bigcup_{i=1}^{\infty} O_i$.

proof) Let $x \in \bigcup_{\lambda \in \Lambda} O_\lambda$. Then $\exists \lambda_x \ni$ $x \in O_{\lambda_x}$. Hence $\exists$ open interval I_{λ_x} with $x \in I_{\lambda_x} \subseteq O_{\lambda_x}$. Take an open subinterval I_x, $x \in I_x \subseteq I_{\lambda_x}$, where I_x has rational endpoints.

Claim $\bigcup_{\lambda \in \Lambda} O_\lambda = \bigcup_{x \in \bigcup_\lambda O_\lambda} I_x$.

(pf) $x \in \bigcup_\lambda O_\lambda \Rightarrow \exists I_x \ni x \in I_x \subseteq \bigcup_x I_x$.
Since $I_x \subseteq O_{\lambda_x}$ for some λ_x, $\bigcup_x I_x \subseteq \bigcup_\lambda O_\lambda$.)

The collection $\mathcal{A} = \{$all open intervals with rational endpoints$\}$ is countable and $\mathcal{D} = \{I_x : x \in \bigcup_\lambda O_\lambda\} \subseteq \mathcal{A}$.

SPRPHD

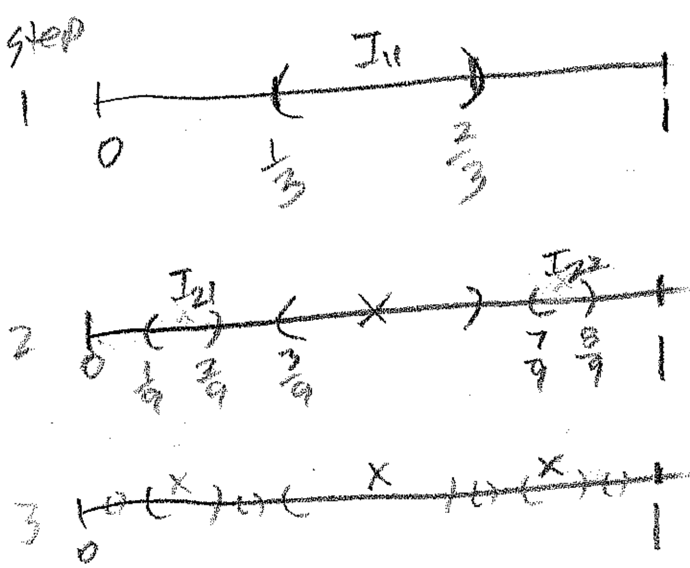
Thus $\mathcal{D}$ is countable; let $\mathcal{D} = \{I_1, I_2, \dots\}$. By construction of the I_i's, $\exists\, \mathcal{O}_\lambda \ni I_i \subseteq \mathcal{O}_\lambda$. Call this $\mathcal{O}_\lambda$ "$\mathcal{O}_i$." Thus

$$\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda = \bigcup_{i=1}^{\infty} I_i \subseteq \bigcup_{i=1}^{\infty} \mathcal{O}_i \subseteq \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda .$$

$$\therefore \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda = \bigcup_{i=1}^{\infty} \mathcal{O}_i . \qquad \square$$

46} * Cor. If a set E can be covered by a collection $\mathcal{C}$ of open sets, then E can be covered by a countable collection of open sets in $\mathcal{C}$.

47} Cantor Set $C \subseteq [0,1]$



# intervals removed		length of removed int's
1	I_{11}, $a_1 = 1$	$\frac{1}{3}$
2, 1	I_{21}, I_{22}, $a_2 = 1$	$\frac{2}{3^2}$
2^2	$I_{31}, I_{32}, I_{33}, I_{34}$, $a_3 = 1$	$\frac{4}{3^3}$

continue.

Let $G = \underbrace{\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{nk}}_{\text{countable union of open intervals}}$. G is open. prop 2.7

The <u>cantor set</u> $C = [0,1] - G = \underbrace{[0,1] \cap G^c}_{\text{closed}}$

$\therefore$ <u>C is closed.</u>

Length of $G = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n =$

$\frac{1}{2} \frac{\frac{2}{3}}{\underbrace{1-2/3}_{1/3}} = \frac{1}{2}\, 2 = 1.$

$\therefore$ <u>"length of $C = 0$."</u> (measure) $\quad \overset{\text{Ch. 3}}{\downarrow}\ m(C) = 0$

It can be shown that C is uncountable, and that there is a 1-1 correspondence between C and $[0,1]$.

((A ternary expansion $0.a_1 a_2 a_3 \ldots$ has $a_i \in \{0,1,2\}$. C consists of the ternary expansions where $a_i \in \{0,2\}$ that do not have $a_i = 0\ \forall i > N$ for some N. So there is a 1-1 correspondence with unique binary expansions.))

(C is not just the endpoints of the removed intervals $\cup \{0,1\}$.)

§2.6 48} Def. Let $f: E \to Y \subseteq \mathbb{R}$

a) f is continuous at point $x \in E$

if $\forall \varepsilon > 0 \ \exists \underset{= \delta_{\varepsilon, x}}{\delta} > 0 \ \ni \forall y \in E$ with

$|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

b) f is continuous on $A \subseteq E$ if f is continuous $\forall x \in A$

c) f is continuous if f is continuous on its domain E.

49} If f is continuous on a closed and bounded set F, then $\exists x_1, x_2 \in F \ni$

$\underbrace{f(x_1)}_{\text{min on } F} \leq f(x) \leq \underbrace{f(x_2)}_{\text{max on } F} \ \forall x \in F$

50} Def. Let $f: E \to Y \subseteq \mathbb{R}$. Then f is uniformly continuous on E if $\forall \varepsilon > 0 \ \exists \underset{\text{ind of } x \text{ and } y \in E}{\delta > 0} \ni$

$\forall x, y \in E$ with $|x-y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

51} A sequence of functions f_n defined on E converges pointwise on E to a function f if

$\forall x \in E$, $\lim_{n \to \infty} f_n(x) = f(x)$. Thus $\forall x \in E$ and

$\forall \varepsilon > 0$, $\exists \underset{= N_{x, \varepsilon}}{N} \ni \forall n \geq N$, $|f(x) - f_n(x)| < \varepsilon$.

($E \subseteq$ domain f_n for $n = 1, 2, \ldots$. Define $f: E \to \mathbb{R}$ by $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in E$.)

52} Def. A sequence of functions f_n defined on E (501 28)
converges uniformly on E to f if $\forall \varepsilon > 0$ $\exists N$ $\ni$ $\forall n \geq N$ and $\forall x \in E$, $|f(x) - f_n(x)| < \varepsilon$.
$N = N_\varepsilon$ is free of x

§2.7 again

53} The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ is generated by the class $\mathcal{C}_0$ of all open sets in $\mathbb{R}$. Thus

$$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{C}_0) = \sigma(\{O : O \subseteq \mathbb{R} \text{ is open}\}).$$

Fact $\mathcal{B}(\mathbb{R}) = \sigma(\{A : A \subseteq \mathbb{R} \text{ is closed}\}) = \sigma(\mathcal{C}_c)$

$= \sigma(\{A : A \subseteq \mathbb{R} \text{ is an open interval}\}) = \sigma(\mathcal{C}_{oI})$

$= \sigma(\{A : A \subseteq \mathbb{R} \text{ is a closed interval}\}) = \sigma(\mathcal{C}_{cI})$

see back of HW4

know
prove $\sigma(\mathcal{C}_0) = \sigma(\mathcal{C}_c)$.

($\subseteq$)

Suppose $A \in \mathcal{C}_c$ so A is a closed set.

Then A^c is open $\Rightarrow A^c \in \sigma(\mathcal{C}_0)$

Thus $A = (A^c)^c \in \sigma(\mathcal{C}_0)$. $\therefore \mathcal{C}_c \subseteq \sigma(\mathcal{C}_0)$.

$$\therefore \underbrace{\sigma(\mathcal{C}_c)}_{\text{smallest } \sigma\text{-algebra containing } \mathcal{C}_c} \subseteq \underbrace{\sigma(\mathcal{C}_0)}_{\text{a } \sigma \text{ algebra containing } \mathcal{C}_c}$$

$(= \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda, \ \mathcal{F}_\lambda \text{ a } \sigma \text{ alg containing } \mathcal{C}_c)$

($\Rightarrow$) Suppose $O \in \mathcal{C}_0$. Then O^c is closed

$\Rightarrow O^c \in \sigma(\mathcal{C}_c) \Rightarrow O = (O^c)^c \in \sigma(\mathcal{C}_c)$.

$\therefore \mathcal{C}_0 \subseteq \sigma(\mathcal{C}_c)$,

$\therefore \sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{C}_c)$

$\square$

open intervals

HW4

know

Prove $\sigma(\mathcal{C}_0) = \sigma(\mathcal{C}_{OI})$

($\Rightarrow$) Let $O \in \mathcal{C}_0$. Then $O = \bigcup_{i=1}^{\infty} (a_i, b_i)$

by prop 2.8. $\therefore O \in \sigma(\mathcal{C}_{OI})$

$\therefore \mathcal{C}_0 \subseteq \sigma(\mathcal{C}_{OI})$

$\therefore \sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{C}_{OI})$

($\Leftarrow$) $\mathcal{C}_{OI} \subseteq \mathcal{C}_0$

class of open intervals class of open sets

$\therefore \mathcal{C}_{OI} \subseteq \sigma(\mathcal{C}_0)$

$\therefore \sigma(\mathcal{C}_{OI}) \subseteq \sigma(\mathcal{C}_0)$

$\square$

Proof technique! If $\mathcal{C} \subseteq$ σ-algebra $\mathcal{F}$

then $\sigma(\mathcal{C}) \subseteq \mathcal{F}$

End Exam 1 material

Ch 1 and 2 key words

exam 1 memorization

what you want to memorize

28.5

σ-algebra

Nonempty class $\mathcal{F}$ of subsets of X ∋ $A_1, A_2 \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

$\mathbb{B}_{[0,1]}$

$\sigma(\mathcal{C})$, $\mathcal{C}$ = class of all open sets in $[0,1]$

$(= \sigma(\mathcal{C}_c) = \sigma(\mathcal{C}_{(a,b]}) = \sigma(\mathcal{C}_{(a,b)})$ etc.$)$

Demorgan's Laws:

$$\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$$

$$\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$$

and $[\bigcap \ldots \bigcup A_\lambda^c]^c =$ $\bigcup \ldots \bigcap A_\lambda$

---

inverse image of $\mathbb{B}$

$f^{-1}[\mathbb{B}] = \{x \in X : f(x) \in \mathbb{B}\}$

$\mathcal{F} = \bigcap_\lambda \mathcal{F}_\lambda$ is a σ-alg

$A, A_1, A_2 \ldots \in \mathcal{F}$

$\Rightarrow A, A_1, A_2 \ldots \in \mathcal{F}_\lambda \quad \forall \lambda$

$\Rightarrow A^c \in \mathcal{F}_\lambda \quad \forall \lambda$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda \quad \forall \lambda$

$\therefore A^c \in \mathcal{F}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$\mathcal{F} = \sigma(A) = \bigcap_{\lambda \in \Lambda} \mathcal{F}_\lambda$

also show Λ is nonempty since $\underbrace{P(X)}_{\text{all subsets of } X} \in \Lambda$

$\sigma(\mathcal{C}_o) = \sigma(\mathcal{C}_c)$

show $\mathcal{C}_o \subseteq \sigma(\mathcal{C}_c)$

so $\sigma(\mathcal{C}_o) \subseteq \sigma(\mathcal{C}_c)$

show $\mathcal{C}_c \subseteq \sigma(\mathcal{C}_o)$

so $\sigma(\mathcal{C}_c) \subseteq \sigma(\mathcal{C}_o)$

Ch 3 Lebesgue measure

§3.1 1] want measure $m(E)$ to be defined on a σ-algebra $\mathfrak{M}$ of subsets of X.

2) Let $P(X)$ = all subsets of X. This σ-algebra is usually too large. $\mathcal{B}([0,1])$ and $\mathcal{B}(\mathbb{R})$ work for $X = [0,1]$ or $X = \mathbb{R}$.

3) $m(E) = \infty$ will be allowed.

4) Let $\ell(I)$ = length of interval I.

ex] If $I = (a,b)$, $[a,b]$, $(a,b]$ or $[a,b)$, then $\ell(I) = b-a \geq 0$.

5] Fact i) $\ell(I) \geq 0$

ii) $\ell\left(\bigcup_{i=1}^{\infty} I_i\right) \leq \sum_{i=1}^{\infty} \ell(I_i)$.

§3.2

6} Notation: Let $A \subseteq \mathbb{R}$ and $I \subseteq \mathbb{R}$ be understood. Let measure = Lebesgue measure unless stated otherwise, m for Lebesgue, μ for general measure

7} Let A be a set and let $\mathcal{C}(A)$ be countable collections $\{I_n\}$ of open intervals that contain A.

8} *Def. Let A be a set and let

$$\mathcal{C}(A) = \left\{ \{I_n\}_{n=1}^{N},\ N \le \infty,\ A \subseteq \bigcup_{n=1}^{N} I_n,\ I_n \text{ an open interval } \forall n \in \mathbb{N} \right\}.$$

Then the <u>(Lebesgue) outer measure</u> of a set A is

$$m^*(A) = \inf_{\mathcal{C}(A)} \sum_{n=1}^{\infty} \ell(I_n)$$

9} Remarks i) Is this definition well defined?

A] Yes. Each $\ell(I_n) \ge 0$. $\therefore$ $\sum_{n=1}^{\infty} \ell(I_n)$ has a lower bound 0 (zero) for any $\{I_n\}_{n=1}^{\infty} \in \mathcal{C}(A)$. (Take $I_n = \phi = (0,0)$ for $n > N$.)

ii) m^* is a set function with domain $P(\mathbb{R})$ = all subsets of $\mathbb{R}$.

$$m^*: P(\mathbb{R}) \to [0,\infty) \cup \{\infty\}$$

$$\text{"} = [0,\infty] \text{"}$$

iii) why are the coverings restricted to be countable?

A} Any uncountable open covering can be written as a countable covering by the Lindelof covering th.

$$\text{So } \bigcup_{\lambda\in\Lambda} I_\lambda = \bigcup_{i=1}^{\infty} I_i$$

iv) know $m^*(\phi) = 0$

proof: Let $I_n = (-\frac{1}{n}, \frac{1}{n})$ and $N=1$.

Then $A = \phi \subseteq I_n \quad \forall n \in \mathbb{N}$.

Thus $m^*(\phi) \le \ell(-\frac{1}{n}, \frac{1}{n}) = \frac{2}{n} \quad \forall n \in \mathbb{N}$

$\therefore 0 \le m^*(\phi) < \delta$ for any $\delta > 0$.

$\therefore m^*(\phi) = 0$.

v) know Monotonicity! Let $A \subseteq B$.

Then $m^*(A) \leq m^*(B)$.

proof} Let $\mathcal{C}(A) = \{ \{I_n\}_{n=1}^{\infty}, A \subseteq \bigcup_{i=1}^{N} I_n,$

I_n open, $N \leq \infty\}$. Let

$\mathcal{C}(B) = \{ \{J_k\}_{k=1}^{M}, B \subseteq \bigcup_{k=1}^{M} J_k, J_k$ open, $M \leq \infty\}$.

Done if $A = B$. Assume $A \neq B$.

Every cover of B is a cover of A.

$\therefore \mathcal{C}(B) \subseteq \mathcal{C}(A)$. Thus

$$m^*(B) = \underbrace{\inf_{\mathcal{C}(B)}}_{\text{smaller subset}} \sum_{k=1}^{M} \ell(J_k) \geq \inf_{\mathcal{C}(A)} \sum_{i=1}^{N} \ell(I_i) = m^*(A)$$

so inf is bigger $\square$

10} know Prop 3.1. The outer measure of an interval is its length.

case 1} Assume $\ell(I) < \infty$ is bounded.

a) First assume $I = (a,b)$.

Claim $m^*(I) = b-a$.

By def of m^*,

$$m^*(I) = \inf_{\mathcal{C}(I)} \sum_n \ell(I_n) \underset{\substack{\uparrow \\ I \subseteq I \text{ open}}}{\leq} \ell(I) = b-a$$

Now $\underset{\substack{\uparrow \\ \text{by def of inf}}}{\exists}$ $\{I_n\}_{n=1}^{N} \in \mathcal{C}(I)$

such that $\sum_{n=1}^{N} \ell(I_n) \leq m^*(I) + \varepsilon_N$

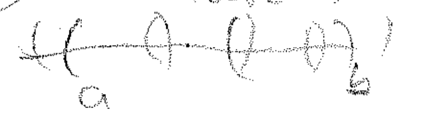
where $\varepsilon_N = \varepsilon(N, \{I_n\}_{n=1}^{N}) > 0$ can be chosen arbitrarily small.

can't let $\varepsilon_N \to 0$ since ε_N depends on $\{I_n\}$.

Now for any $\{I_n\}_{n=1}^{N} \in \mathcal{C}(I)$, $I \subseteq \bigcup_{n=1}^{N} I_n$.

$\therefore$ $\ell(I) \leq \sum_{n=1}^{N} \ell(I_n) \quad \forall \{I_n\}_{n=1}^{N} \in \mathcal{C}(I)$

there could be overlapping I_n



Hence $\ell(I) \leq m^*(I) + \varepsilon \quad \forall \varepsilon > 0$.

(regardless of N and $\{I_n\}_{n=1}^{N} \in \mathcal{C}(I)$,

ie ε does not depend on N and $\{I_n\}_{n=1}^{\infty}$.)

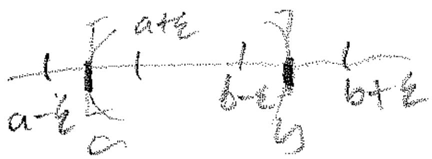
Let $\varepsilon \to 0$ to show $b-a = \ell(I) \leq m^*(I)$.

$\therefore$ $m^*(I) \leq b-a = \ell(I) \leq m^*$

so $m^*(I) = b-a$.

b) Now consider any finite interval with endpoints a and b.

For any $\varepsilon > 0$ $\underbrace{(a+\varepsilon, b-\varepsilon)}_{\phi \text{ if } a+\varepsilon \geq b-\varepsilon \text{ so}} \subseteq I \subseteq (a-\varepsilon, b+\varepsilon)$



By monotonicity of m^*, if $0 < 2\varepsilon < b-a$,

then $b-a-2\varepsilon = m^*(a+\varepsilon, b-\varepsilon) \leq m^*(I) \leq$

$m^*(a-\varepsilon, b+\varepsilon) = b-a+2\varepsilon$ by case 1 a).

Since this is true $\forall \varepsilon > 0$, we can let $\varepsilon \to 0$

to get $b-a \leq m^*(I) \leq b-a$.

$\therefore$ $b-a = m^*(I) = \ell(I)$.

Case 2] $\ell(I) = \infty$.

Need to prove $m^*(I) = \infty$.

For any $\Delta > 0$, $\exists$ bounded interval J

$\ni \ell(J) = \Delta$ and $J \subseteq I$. By case 1

and monotonicity, $m^*(I) \geq m^*(J) = \ell(J) = \Delta$.

Since this inequality is true for any $\Delta > 0$,