let $\Delta \to \infty$. $\therefore$ $m^*(I) = \infty$. $\square$

ex} * $m^*(\{a\}) = m^*([a,a]) = 0$ for any singleton.

11} $\overset{\text{know}}{\text{Prop 3.2}}$ <u>countable subadditivity</u>:

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets.

Then $m^*(\underbrace{\bigcup_{n=1}^{\infty} A_n}_{B}) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

case 1} If at least one $m^*(A_n) = \infty$,

then the inequality holds.

(since RHS $= \infty$ and $m^*(A) \leq \infty$ for any $A \in X$.)

case 2} Suppose $m^*(A_n) < \infty$ $\forall n \in \mathbb{N}$.

For given $\varepsilon > 0$ and for any $n \in \mathbb{N}$,

by def of m^*, $\exists$ $\{I_{ni}\}_{i=1}^{N_n}$ $\ni$

$$\sum_{i=1}^{N_n} l(I_{ni}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$$

by def of inf

where $\{I_{ni}\}_{i=1}^{N_n} \in \mathcal{C}(A_n)$

((so $A_n \subseteq \bigcup_{i=1}^{N_n} I_{ni}$, I_{ni} open intervals))

$$(*) \quad \therefore \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \ell(I_{ni}) \le \sum_{n=1}^{\infty} m^*(A_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}.$$

Since $\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{N_n} I_{ni}$,

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \overset{\text{monotonicity}}{\le} m^*\left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{N_n} I_{ni}\right)$$

$$\underset{\text{def of } m^*}{\le} \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} \ell(I_{ni}) \underset{\text{by } (*)}{\le} \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon$$

def of m^*
let $\{I_{ni}\}_{n,i=1}^{\infty} = \{I_j\}_{j=1}^{\infty}$

and $\underbrace{\{I_{ni}\}_{n,i=1}^{\infty}}$ $\subseteq \mathcal{C}\left(\bigcup_{n=1}^{\infty} A_n\right)$

(the I_{ni}'s in both sums relabel to get one sequence)

Let $\varepsilon \to 0$ to get the inequality.

$\square$

ex] I know $m^*\left(\bigcup_{n=1}^{N} A_n\right) \le \sum_{n=1}^{N} m^*(A_n)$

(Finite Subadditivity)

Proof take $A_n = \phi$ for $n > N$.

$\square$

12} know ~~p58~~ cor. If set E is countable, then $m^*(E) = 0$.

proof} Let $E = \{x_i\}_{i=1}^{\infty}$.

$$m^*(E) = m^*\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) \le \sum_{i=1}^{\infty} m^*(\{x_i\})$$

$$= \sum_{i=1}^{\infty} 0 = 0$$ since $\{x_i\} = [x_i, x_i]$.

$\square$

13}* Cor: The set $[0,1]$ is not countable.

proof $m^*[0,1] = 1 > 0$.

$\square$

(If $A = [0,1]$ is countable, then $m^*(A) = 0$.)

$\therefore$ A is not countable.

14} P.53 Def. A set $G \in G_\delta$ if G is a countable intersection of open sets.

15} p58 prop 3.5 (corrected)

a) Given any set A and any $\varepsilon > 0$, there is an open set O such that

$A \subseteq \mathcal{O}$ and $m^*(\mathcal{O}) \leq m^*(A) + \varepsilon$.

b) There is $G \in G_\delta$ such that
$A \subseteq G$ and $m^*(G) = m^*(A)$.

proof a) If $m^*(A) = \infty$ take $\mathcal{O} = \mathbb{R}$.
suppose $m^*(A) < \infty$. Then
$\forall \varepsilon > 0$, $\exists \{I_n\}_{n=1}^{N} \in \mathcal{C}(A)$, $N \leq \infty$
$\ni \sum_{n=1}^{N} \ell(I_n) \leq m^*(A) + \varepsilon$.

Take $\mathcal{O} = \bigcup_{n=1}^{\infty} I_n$. Then $A \subseteq \mathcal{O}$
since $\{I_n\}_{n=1}^{N} \in \mathcal{C}(A)$.

$\mathcal{O}$ is open and

$$m^*(\mathcal{O}) \underset{\substack{\uparrow \\ \text{subadditivity}}}{\leq} \sum_{i=1}^{N} m^*(I_n) = \sum_{i=1}^{N} \ell(I_n) \leq m^*(A) + \varepsilon.$$

b) For any $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$,
$\exists \mathcal{O}_k \ni m^*(\mathcal{O}_k) \leq m^*(A) + \frac{1}{k}$. (by a)

Take $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$. Then
i) $G \in G_\delta$, and since $A \subseteq \mathcal{O}_k$ $\forall k$
ii) $A \subseteq G$.

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iii) $m^*(A) \le m^*(G) \le m^*(O_k) \le m^*(A) + \frac{1}{k}$.

This statement is true $\forall k \in \mathbb{N}$.

Let $k \to \infty$ (take lim inf of each term).

Then $m^*(A) \le m^*(G) \le m^*(A)$.

Thus $m^*(A) = m^*(G)$

$\square$

§3.3

16] A set function μ is <u>countably additive</u> if $\mu : \underbrace{\mathcal{F}}_{\sigma\text{-algebra of subsets of } X} \to [0, \mu(X)]$

such that $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$ if

$E_1, E_2, \ldots$ are <u>disjoint</u> sets in $\mathcal{F}$.

17] m^* is not countably additive on $\mathcal{F} = \mathcal{P}(\mathbb{R}) =$ all subsets of $\mathbb{R}$.

18] Analysts have shown that general measures μ, including m, should satisfy 16] on the σ-algebra $\mathcal{F}_m$ of measurable sets, and on $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}_m$.

19] Know p58 Def. A set E is (Lebesgue) <u>measureable</u> if for any set A $(\subseteq X = \mathbb{R})$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

disjoint

20] Know E is measurable iff E^c is measurable. $\left(m^*(A \cap E^c) + m^*(A \cap E^{c^c}) = m^*(A \cap E^c) + m^*(A \cap E) \right)$

ex] i) Let $E = \mathbb{R}$. Then

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \phi) = m^*(A) + m^*(\phi) = m^*(A).$$

ii) Let $E = \phi$. Then

$$m^*(A) = m^*(A \cap \phi) + m^*(A \cap \mathbb{R}) = m^*(\phi) + m^*(A) = m^*(A).$$

Know Thus $\mathbb{R}$ and ϕ are measurable.

21] Think of the sets A as test sets. Some texts replace A by T. Think of measurable sets as good sets. Almost any set $E \subseteq \mathbb{R}$ that you

can imagine) before taking   501 35
measure th, is measurable.

22} p58 If $m^*(E)=0$, then E is measurable
and $m(E)=0$.

Proof: Let $A \subseteq \mathbb{R}$.
Then $A\cap E \subseteq E$ $\therefore$ $m^*(A\cap E) \le m^*(E)=0$
$\therefore$ $m^*(A\cap E)=0$.
Similarly $A\cap E^c \subseteq A$.
Thus $m^*(A\cap E^c) \le m^*(A)$
$\therefore$ $m^*(A\cap E^c)+m^*(A\cap E) \le m^*(A)$
$\square$ by 23}

23} E is measurable iff for each $A \subseteq \mathbb{R}$,
$m^*(A\cap E)+m^*(A\cap E^c) \le m^*(A)$.

proof} $A=(A\cap E)\cup(A\cap E^c)$
$\therefore$ $m^*(A) \overset{\text{finite subadditivity}}{\le} m^*(A\cap E)+m^*(A\cap E^c)$
$\therefore$ if 23} holds then $m^*(A)=m^*(A\cap E)+m^*(A\cap E^c)$
$\square$

24}* Let $\mathfrak{M}=\mathfrak{F}_m$ be the class of
measurable sets. $\therefore$ E is measurable
iff $E\in\mathfrak{F}_m$.

25] If $E_1, E_2 \in \mathfrak{F}_m$, then $E_1 \cup E_2 \in \mathfrak{F}_m$.

proof] By 23] need to show $\forall A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

$$(*) \quad \underset{\text{DeMorgan}}{=} m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1^c \cap E_2^c)].$$

Since $E_2 \in \mathfrak{F}_m$, take test set $\overbrace{A \cap E_1^c}^{B \text{ say}}$.

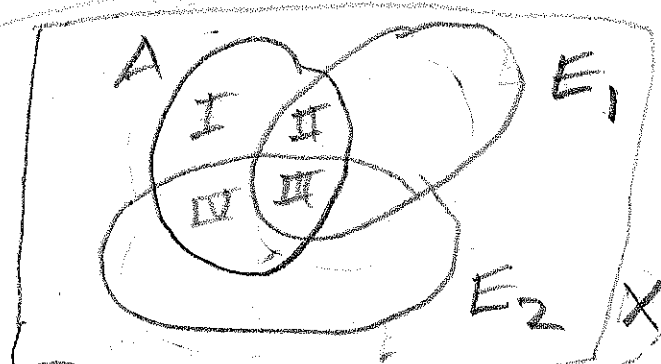
Then $m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$.

Now $A \cap (E_1 \cup E_2) = \overset{\mathrm{II} \cup \mathrm{III}}{(A \cap E_1)} \cup \overset{\mathrm{III} \cup \mathrm{IV}}{(A \cap E_2)} =$

$\underset{\mathrm{II} \cup \mathrm{III}}{(A \cap E_1)} \cup \underset{\mathrm{IV}}{(A \cap E_2 \cap E_1^c)}$.

LHS = $\mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$

RHS = $\mathrm{II} \cup \mathrm{III} \cup \mathrm{IV}$



(**) Thus $m^*[A \cap (E_1 \cup E_2)] \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$.

so $m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1^c \cap E_2^c)] = (*)$

$$\underset{\text{by}(**)}{\leq} m^*(A \cap E_1) + m^*[A \cap E_2 \cap E_1^c] + m^*[A \cap E_1^c \cap E_2^c]$$

$$\underset{E_2 \in \mathfrak{F}_m}{=} m^*(A \cap E_1) + m^*(A \cap E_1^c) \underset{E_1 \in \mathfrak{F}_m}{=} m^*(A)$$

$\square$

26) $A, B \in \mathfrak{F}_m \Rightarrow$

$\underbrace{A^c \in \mathfrak{F}_m}_{\text{by 20)}}$ and $\underbrace{A \cup B \in \mathfrak{F}_m}_{\text{by 25)}}$

$\therefore$ $\mathfrak{F}_m$ is an algebra ($\mathbb{R} \in \mathfrak{F}_m$ by ex. under 20))

27) Suppose $E_1, \dots, E_n \in \mathfrak{F}_m$ are disjoint.

Then $m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i)$.

($\underline{\text{Finite additivity if } E_i \in \mathfrak{F}_m.}$)

proof) $\bigcup_{i=1}^{n} E_i \in \mathfrak{F}_m$ since $\mathfrak{F}_m$ is an algebra.

So need to show $m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$.

use induction!

If $n=1$, $m^*(E_1) = m^*(E_1)$.

suppose $n \le k-1$ and $m^*\left(\bigcup_{i=1}^{k-1} E_i\right) = \sum_{i=1}^{k-1} m^*(E_i)$.

If $n=k$, want to show $m^*\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m^*(E_i)$.

$E_k \in \mathfrak{F}_m$ and let test set $= \bigcup_{i=1}^{k} E_i$.

Then $m^*\left(\bigcup_{i=1}^{k} E_i\right) = m^*\Big(\underbrace{\bigcup_{i=1}^{k} E_i \cap E_k}_{E_k}\Big) + m^*\Big(\underbrace{\bigcup_{i=1}^{k} E_i \cap E_k^c}_{\bigcup_{i=1}^{k-1} E_i \text{ since the } E_i \text{ are disjoint}}\Big)$

$= m^*(E_k) + m^*\left(\bigcup_{i=1}^{k-1} E_i\right) = m^*(E_k) + \sum_{i=1}^{k-1} m^*(E_i)$

$\square$

Note: Let $E_1, E_2, \ldots \in \mathfrak{M}$, $\underbrace{E_i \cap E_j = \emptyset}_{\text{disjoint}}$ for $i \neq j$.

Then for each $n \in \mathbb{N}$,

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i) \quad \text{and}$$

$$\lim_{n\to\infty} m^*\left(\bigcup_{i=1}^{n} E_i\right) = \lim_{n\to\infty} \sum_{i=1}^{n} m^*(E_i)$$

$$= \sum_{i=1}^{\infty} m^*(E_i).$$

(partial sums are an increasing seq)

We do not yet know whether we can exchange limit and m^*:

does $\lim_{n\to\infty} m^*\left(\bigcup_{i=1}^{n} E_i\right) = m^*\left(\lim_{n\to\infty} \bigcup_{i=1}^{n} E_i\right)$

$= m^*\left(\bigcup_{i=1}^{\infty} E_i\right)$?

$\left(A_n = \bigcup_{i=1}^{n} E_i \uparrow A = \bigcup_{i=1}^{\infty} E_i\right)$

(countable additivity) if $E_i \in \mathfrak{M}$

28) Let $E_1, E_2, \ldots \in \mathfrak{M}$ with $E_i \cap E_j = \emptyset$, $i \neq j$.

Then $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$

don't yet know if $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{M}$.

Proof) For each $n \in \mathbb{N}$, $\bigcup_{i=1}^{n} E_i \subseteq \bigcup_{i=1}^{\infty} E_i$.

$\therefore m^*\left(\bigcup_{i=1}^{n} E_i\right) \leq m^*\left(\bigcup_{i=1}^{\infty} E_i\right)$ by monotonicity.

By 27) $m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$.

Now let $n \to \infty$ (take limit of both sides).

Then $\sum_{i=1}^{\infty} m^*(E_i) \le m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m^*(E_i)$

countable subadditivity

$\therefore$ $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$

$\square$

29) ~~know~~ Let $E_1, E_2 \ldots \in \mathfrak{F}_m$. Then $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{F}_m$ and $\mathfrak{F}_m$ = class of measurable sets is a σ-algebra.

proof) Need to show $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{F}_m$.

WLOG, assume $E_i \cap E_j = \phi$ for $i \ne j$.

(Otherwise, $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} G_i$ where

$G_i = \underbrace{E_i - \bigcup_{j=1}^{i-1} E_j}$ and $G_i \cap G_j = \phi$ for $i \ne j$.

$i \ne j$. $E_i \cap \bigcap_{j=1}^{i-1} E_j^c$

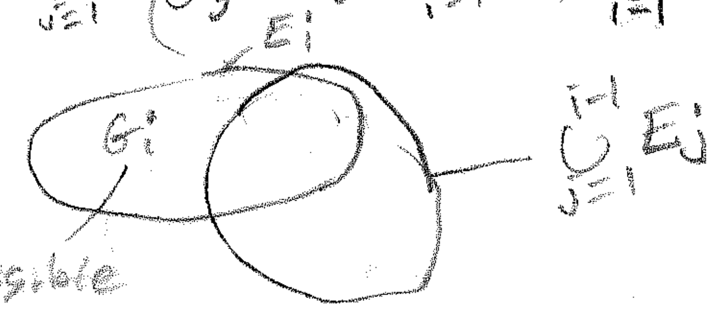
= the part of E_i not in $\bigcup_{j=1}^{i-1} E_j$

$G_1 = E_1$

$G_2 = E_2 - E_1$, ($\bigcup_{i=1}^{2} G_i = \bigcup_{j=1}^{2} E_j$

$G_3 = E_3 - \bigcup_{i=1}^{2} E_i$), $\bigcup_{i=1}^{3} G_i = \bigcup_{j=1}^{3} E_j$)

useful technique



It is enough to prove that for any test set $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*\left(A \cap \left[\bigcup_{i=1}^{\infty} E_i\right]^c\right).$$

Set $E = \bigcup_{i=1}^{\infty} E_i$ and $F_n = \bigcup_{i=1}^{n} E_i \in \mathfrak{M}$.

Since $F_n \in \mathfrak{M}$ for each n,

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c)$$

$$\geq m^*\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] + m^*(A \cap E^c)$$

($F_n \subseteq E \Rightarrow E^c \subseteq F_n^c$, use monotonicity)

$$= \underbrace{\sum_{i=1}^{n} m^*\underbrace{(A \cap E_i)}_{\text{disjoint}}}_{\text{partial sum with nonnegative terms has a limit}} + m^*[A \cap E^c]$$

so take limit of both sides: let $n \to \infty$.

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$$

$$\geq m^*\left[\bigcup_{i=1}^{\infty} (A \cap E_i)\right] + m^*(A \cap E^c)$$

countable subadditivity

$$= m^*(A \cap E) + m^*(A \cap E^c).$$

$\therefore E \in \mathfrak{M}$. $\square$