30} P60 Th. $\underbrace{(a,\infty)}_{E} \in \mathcal{F}_M \quad \forall a \in \mathbb{R}$, 501 38

proof} Let A be any test set.

If $m^*(A) = \infty$, then

$$m^*(A \cap (a,\infty)) + m^*(A \cap \overbrace{(a,\infty)^c}^{(-\infty,a]}) \le m^*(A) = \infty$$

and $A \in \mathcal{F}_M$. for next line without the ∞ want to show.

Suppose $m^*(A) < \infty$.

By def of m^*, $\exists \{I_n\} \in \mathcal{C}(A) \ni$

$$m^*(A) + \varepsilon \ge \sum_n \ell(I_n)$$

$$= \sum_n \left[\ell(I_n \cap (a,\infty)) + \ell(I_n \cap (-\infty,a])\right] \quad (*)$$

$I_n = I_n \cap \mathbb{R} = I_n \cap [(-\infty,a] \cup (a,\infty)]$.

Then use the dist law and the fact that the intersection of 2 intervals is an interval or the empty set. Note that $\ell(I) = m^*(I)$ for an interval.

$$(*) = \sum_n m^*(I_n \cap (a,\infty)) + \sum_n m^*(I_n \cap (-\infty,a])$$

$$\underset{\text{subadditivity}}{\ge} m^*\left[\left(\bigcup_n I_n\right) \cap (a,\infty)\right] + m^*\left(\bigcup_n I_n \cap (-\infty,a]\right)$$

$$\ge m^*[A \cap (a,\infty)] + m^*(A \cap (-\infty,a])$$

Monotonicity with $A \subseteq \bigcup_n I_n$

Thus for each $\varepsilon > 0$,

$$m^*(A) + \varepsilon \geq m^*[A \cap (a, \infty)] + m^*[A \cap (-\infty, a]]$$

Let $\varepsilon \to 0$ to complete the proof. $\square$

31) Th Every Borel set is measurable!

$\mathcal{B}(\mathbb{R}) \subseteq \overline{\mathcal{F}}_M$.

proof) $\overline{\mathcal{F}}_M$ is a σ-algebra. So $\mathbb{R} \in \overline{\mathcal{F}}_M$.

$(a, \infty) \in \overline{\mathcal{F}}_M \;\; \forall a \in \mathbb{R}$ by 30).

Thus $(a, \infty)^c = (-\infty, a] \in \underbrace{\overline{\mathcal{F}}_M}_{\sigma\text{-alg}} \;\; \forall a \in \mathbb{R}$.

So $(-\infty, a) = \bigcup_{n=1}^{\infty} \underbrace{(-\infty, a - \tfrac{1}{n}]}_{\in \overline{\mathcal{F}}_M} \in \overline{\mathcal{F}}_M$

$\therefore (a, b) = (-\infty, b) \cap (a, \infty) \in \overline{\mathcal{F}}_M \;\; \forall a < b \in \mathbb{R}$.

Thus $\mathcal{C}_{OI}$ = class of all open intervals of $\mathbb{R}$ $\subseteq \overline{\mathcal{F}}_M$. $\therefore$ $\sigma(\mathcal{C}_{OI}) = \mathcal{B}(\mathbb{R}) \subseteq \overline{\mathcal{F}}_M$

$\square$

32) Def (know) Let $m(E) = m^*(E)$ if $E \in \overline{\mathcal{F}}_M$.

Then m is the <u>Lebesgue measure</u>.

Note) We usually will replace

$m^*(E)$ by $m(E)$ if $E \in \mathfrak{M}$.

countable Additivity: see 28}

ex} If $E_i \in \mathfrak{M}$ and $E_i \cap E_j = \emptyset$,

then $m\left(\bigcup_i E_i\right) = \sum_i m(E_i)$.

33} Let $A, B \in \mathfrak{M}$ with $A \subseteq B$

and $m(A) < \infty$. Then

$$m(B-A) = m(B) - m(A).$$

Proof} know $B = \underbrace{A \cup (B-A)}_{\text{disjoint } \mathfrak{M} \text{ sets.}}$

$$\therefore\ m(B) = m(A) + m(B-A)$$

$$\therefore\ m(B-A) \underset{\substack{\uparrow \\ \text{since } m(A) < \infty.}}{=} m(B) - m(A)$$

Note: similar to proof for probability.

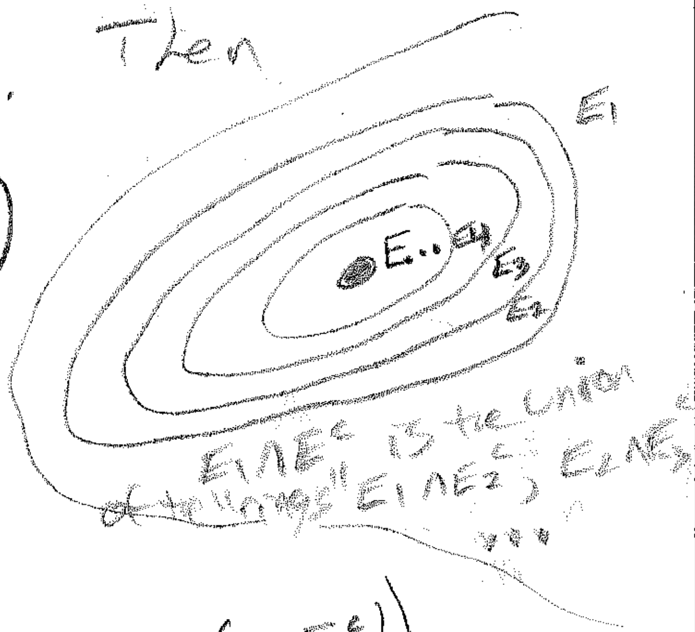
34} Prop 3.14 Let $\{E_n\}$ be an infinite (so $E_n \in \mathfrak{M}$) decreasing sequence of <u>measurable sets</u>,

$E_{n+1} \subseteq E_n$ for each n.

Then $M\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} M(E_n)$.

$E_n \downarrow E$

proof: Let $E = \bigcap_{i=1}^{\infty} E_i$. Then



$$E_1 \setminus E = \bigcup_{i=1}^{\infty} (E_1 \setminus E_i)$$

$$\overset{(*)}{=} \underbrace{\bigcup_{i=1}^{\infty} (E_i - E_{i+1})}_{\text{disjoint union}}$$

$$\left(E_1 \setminus E = E_1 \cap \left(\bigcap_i E_i\right)^c = E_1 \cap \left(\bigcup_i E_i^c\right) = \bigcup_i (E_1 \cap E_i^c) = \bigcup_i (E_1 \setminus E_i) \right)$$

By additivity, $M(E_1 \setminus E)$

$$= \sum_{i=1}^{\infty} M(E_i - E_{i+1}). \quad \text{By } 3.3]$$

$$M(E_1 \setminus E) = M(E_1) - M(E) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[M(E_i) - M(E_{i+1}) \right]$$

nonnegative terms so sum converges on $\mathbb{R}^*$

telescoping series

$$= \lim_{n\to\infty} \left[M(E_1) - M(E_2) + M(E_2) - M(E_3) + \dots + M(E_n) - M(E_{n+1}) \right]$$

$$= m(E_1) - \lim_{n\to\infty} m(E_{n+1}) = m(E_1) - \lim_{n\to\infty} m(E_n).$$

Thus $m(E_1) - m(E) = m(E_1) - \lim_{n\to\infty} m(E_n)$.

Since $m(E_1) < \infty$,

$$m(E) = \lim_{n\to\infty} m(E_n)$$

$\square$

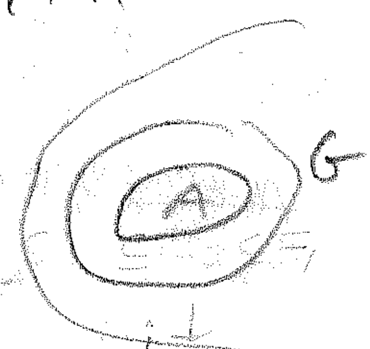
35) Prop. The following are equivalent.

1) $A \in \mathcal{F}_M$

2) $\forall \varepsilon > 0$, $\exists$ open set $O \supseteq A$ with $m(O \setminus A) \le \varepsilon$.

3) $\exists$ G_δ set $G \supseteq A$ with

$$m(G/A) = 0$$



So $G = \underbrace{A}_{\in \mathcal{F}_M} \cup \underbrace{(G-A)}_{\text{measure } 0}$

and $m(G) = m(A)$

36) If $m^*(E) = m(E) = 0$, E is known as a <u>null set</u> or <u>set of measure 0</u>. Null sets

include ϕ, countable sets $A=\{a_1, a_2, \dots\}$, and the uncountable Cantor set.

37} properties of m^*, m, and $\mathcal{F}_M$ are summarized in Exam 2 review 73), 79) and 81).

countable and finite additivity of m on $\mathcal{F}_M$ can be used to prove some HW problems.

§3.4 A non measurable set.

38} $\underline{\text{Know}}$ It can be shown that there are sets $U, U_i \in P(\mathbb{R})$ such that $U, U_i \notin \mathcal{F}_M$. Hence U is not measurable and $U \notin B(\mathbb{R}) \subseteq \mathcal{F}_M$.

Such sets can also be used to show that m^* is not countably additive on $P(\mathbb{R})$.

V = the Vitali set is nonmeasurable.

Takes 2 or 3 lectures.

See HW 6 problem 4.

39] Lebesgue measure on $X = [a,b]$ and $X = [0,\infty)$ is similar.

Note that if $X = [0,1]$, then measurable sets A satisfy

$0 \le m(A) \le m[0,1] = 1$.

(The $U[0,1]$ probability measure is the Leb measure restricted to $\mathcal{B}([0,1])$, roughly.)

§3.5 measurable functions

40] Def. An extended real valued function $f: D \to \mathbb{R}^*$ is a (Leb) measurable function if its domain D is measurable and if for each real number a, the set $f^{-1}[(a,\infty)] = \{x \in D : f(x) \in (a,\infty)\}$ $= \{x \in D : f(x) > a\}$ is measurable.

know

(Could say f is a measurable function on D)

41} A common alternative def is that an extended real valued function $f: D \to \mathbb{R}^*$ is Lebesgue measurable if D is measurable and if for each Borel set $B \in \mathcal{B}(\mathbb{R})$, $f^{-1}(B)$ is (Leb.) measurable.

($\mathcal{B}(\mathbb{R})$ is easier to use than $\mathcal{F}_M$, and the difference between $\mathcal{B}(\mathbb{R})$ and $\mathcal{F}_M$ is sets ~~of~~ measure 0.)

$$f^{-1}(B) = \{x \in D ; f(x) \in B\}$$

42} From exam 1 rev 7) $f: D \to Y$ behavior is not as nice.

$$f^{-1}\left[\bigcup_i B_i\right] = \bigcup_i f^{-1}[B_i], \quad f^{-1}\left[\bigcap_i B_i\right] = \bigcap_i f^{-1}[B_i]$$

and $f^{-1}[A^c] = [f^{-1}(A)]^c$ if $f: X \to Y$.

43) Real valued functions are also extended real valued functions.

44} Th. An extended real valued function $f: D \to \mathbb{R}^*$ is a (Leb) measurable function if D is measurable and iff any one of the following 3 conditions hold.