501 42

a) $\forall a \in \mathbb{R}$, $f^{-1}[[a,\infty]] = \{x \in D : f(x) \ge a\}$ is a measurable set

b) $\forall a\in\mathbb{R}$ $f^{-1}[[-\infty,a)] = \{x\in D: f(x)<a\}$ is a measurable set.

c) $\forall a \in \mathbb{R}$ $f^{-1}[[-\infty,a]] = \{x\in D: f(x)\le a\}$ is a measurable set. Then

know

Proof} $\{x\in D: f(x)>a\} = \underbrace{D - \{x\in D: f(x)\le a\}}_{\{x: f(x)\le a\}^c \text{ if } D=\mathbb{R}}$

so c) holds. Now

$\{x\in D: f(x)\ge a\} = D - \{x\in D: f(x)<a\}$.

So a) holds iff b) holds. If f is a measurable function, then $\forall a\in\mathbb{R}$,

$\{x\in D: f(x)>a\}$ is measurable by def.

Then $\{x\in D: f(x)>a-\frac{1}{n}\}$ is measurable $\in \mathfrak{M}$

for $n=1,2,\ldots,$

$$\therefore \bigcap_{n=1}^{\infty} \{x\in D: f(x)>a-\tfrac{1}{n}\} = \{x\in D: f(x)\ge a\}$$

is measurable.

$$\left(\bigcap_{n=1}^{\infty} (a-\tfrac{1}{n} < y <\infty) \neq \bigcap_{n=1}^{\infty}\{y>a-\tfrac{1}{n}\} = \{y\ge a\} \neq \{a\le y<\infty\}\right)$$

Conversely, if $\{x \in D : f(x) \geq a\}$ is measurable $\forall a \in \mathbb{R}$, then

$\{x \in D : f(x) \geq a + \frac{1}{n}\}$ is measurable

for each $n \in \mathbb{N}$. $\therefore$

$$\bigcup_{n=1}^{\infty} \{x \in D : f(x) \geq a + \tfrac{1}{n}\} = \{x \in D : f(x) > a\}$$

is measurable for each $a \in \mathbb{R}$.

$\square$

(like $\bigcup_{n=1}^{\infty} [a + \frac{1}{n}, \infty) = (a, \infty)$)

45} Almost any function you can think of before taking a measure theory class is a measurable function.

46}* Let the characteristic function (analysis) $\leftarrow$ prob and stat

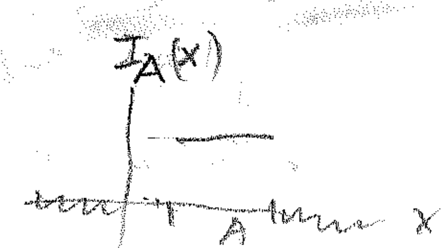
= indicator function = (look to similar)

$$\chi_A(x) = I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

$\uparrow$ chi

U[illegible] take $I_A : \underbrace{\mathbb{R}}_{\text{or } X} \to \{0, 1\}$.

47) Th. $I_A(x)$ is a measurable function iff A is measurable.
(Assume $I_A : \mathbb{R} \to \{0,1\}$.)



know

proof: $I_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$

$I_A(x) \le 1$

$$\{x : I_A(x) > \alpha\} = \begin{cases} \phi & \text{if } \alpha \ge 1 \\ A & \text{if } 0 \le \alpha < 1 \\ \mathbb{R} & \text{if } -\infty < \alpha < 0 \end{cases}$$

For $x \in A$, $I_A(x) = 1 > \alpha$

$I_A(x) \in \{0,1\} > \alpha \ \forall x \in \mathbb{R}$

$\mathbb{R} \in \mathcal{F}_M$, $\phi \in \mathcal{F}_M$, $A \in \mathcal{F}_M$, iff A is measurable

$\square$

Note: If $f : D \xrightarrow{\text{measurable}} \{0,1\}$, then f is a measurable function iff $A \cap D$ is measurable.

$$\{x : I_A(x) > \alpha\} = \begin{cases} \phi & \text{if } \alpha \ge 1 \\ A \cap D & \text{if } 0 \le \alpha < 1 \\ D & -\infty < \alpha < 0 \end{cases}$$

48)* A nonmeasurable function is $I_V(x)$ where V is a nonmeasurable set such as the Vitali set.

phi

49} The function φ is a <u>simple function</u>

if $\varphi(x) = \sum_{i=1}^{n} \alpha_i \, I_{A_i}(x)$ where

$A_i \subseteq \mathbb{R}$, $\alpha_i \in \mathbb{R}$ and each $A_i \in \mathfrak{F}_M$

∴ simple functions are measurable.

50} A <u>step function</u> is a simple function where all A_i are intervals.

intervals $\in \mathfrak{F}_M$.

$\psi(x) = \sum_{i=1}^{n} \alpha_i \, I_{A_i}(x)$.

psi

D is a measurable set by the def of a measurable function

51} Let $\mathcal{L}(D) = \{$ all real valued L. measurable functions with domain D $\}$

Let $\overline{\mathcal{L}}(D) = \{$ all extended real valued L. measurable functions with domain D $\}$.

52} Let $f \in \overline{\mathcal{L}}(D)$. Then for any $\alpha \in \overbrace{\mathbb{R} \cup \{\pm\infty\}}^{\mathbb{R}^*}$, the set $\{x \in D : f(x) = \alpha\} \in \mathfrak{F}_M$.

proof: Case 1) $\alpha \in \mathbb{R}$:

$\{x \in D : f(x) = \alpha\} = \{x \in D : f(x) \leq \alpha\} \cap \{x \in D : f(x) \geq \alpha\}$
$\in \mathfrak{F}_M$

case 2} $\alpha = \infty$

$$\{x \in D : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) \geq n\} \in \mathfrak{F}_M.$$

case 3} $\alpha = -\infty$

$$\{x \in D : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \underbrace{\{x \in D : f(x) < -n\}}_{\in \mathfrak{F}_M \text{ since } f \text{ is measurable}} \overset{\sigma\text{-alg}}{\in \mathfrak{F}_M}$$

53} Thm Let $c \in \mathbb{R}$, $f \in \mathcal{S}(D)$ and $g \in \mathcal{S}(D)$. Then

i) $c+f \in \mathcal{S}(D)$

ii) $f+g \in \mathcal{S}(D)$

iii) $f-g \in \mathcal{S}(D)$

iv) $f \cdot g \in \mathcal{S}(D)$ $\leftarrow f(x)g(x)$ not $f \circ g = f(g(x))$

v) If $f(x) \neq 0$ $\forall$ $x \in D$, then
$\frac{1}{f} \in \mathcal{S}(D)$,

vi) $cf \in \mathcal{S}(D)$.

Proof (except for v)) i) know For any $\alpha \in \mathbb{R}$, $\{x \in D : f(x)+c > \alpha\}$

$= \{x \in D : f(x) > \alpha - c\} \in \mathfrak{F}_M$ since $\underbrace{f \in \mathcal{S}(D)}_{f \text{ is measurable}}$.

ii) $\{x\in D : f(x)+g(x)>\alpha\} =$

$\bigcup_{r\in\mathbb{Q}}\left[\{x\in D : f(x)>r\}\cap\{x\in D : g(x)>\alpha-r\}\right]\in\mathfrak{F}_M.$

(For any $x_0\in LHS$ $\exists$ $r_0\in\mathbb{Q}$ $\ni$

$f(x_0)>r_0>\alpha-g(x_0)$

$f(x_0)>r_0$ and $g(x_0)>\alpha-r_0$

$\therefore$ $x_0\in RHS$. So $LHS\subseteq RHS$.

For any $x_0\in RHS$, $\exists$ $r_0\in\mathbb{Q}$ $\ni$

$x_0\in\{x\in D : f(x)>r_0\}\cap\{x\in D : g(x)>\alpha-r_0\}$

$\Rightarrow f(x_0)>r_0>\alpha-g(x_0)$

$\Rightarrow f(x_0)>\alpha-g(x_0)$

$\Rightarrow x_0\in\{x\in D : f(x)>\alpha-g(x)\}$

$\therefore$ $x_0\in LHS$ and $RHS\subseteq LHS$

vi) know case 1) $c>0$:

$\{x\in D : cf(x)>\alpha\} = \{x\in D : f(x)>\frac{\alpha}{c}\}\in\mathfrak{F}_M$

case 2) $c<0$:

$\{x\in D : cf(x)>\alpha\} = \{x\in D : f(x)<\frac{\alpha}{c}\}\in\mathfrak{F}_M$

Sol 45

case 3 $c = 0$:

$$\{x \in D : \underbrace{cf(x)}_{0} > \alpha\} = \begin{cases} \phi, & \alpha \geq 0 \\ D & \alpha < 0 \end{cases}$$

and $\phi, D \in \mathfrak{F}_m$

iii) know since $g \in \mathcal{L}(D)$

$-g \in \mathcal{L}(D)$ by vi) with $c = -1$.

$\therefore f + (-g) = f - g \in \mathcal{L}(D)$ by ii)

iv) claim: $f \in \mathcal{L}(D) \Rightarrow f^2 \in \mathcal{L}(D)$.

pf) For any $\alpha \in \mathbb{R}$,

$\{x \in D : [f(x)]^2 > \alpha\}$

0 may not > 0 but $[f(x)]^2 \geq 0$, so true $\forall x \in D$ for $\alpha < 0$

$$= \begin{cases} D & \alpha < 0 \\ \{x \in D : f(x) > \sqrt{\alpha}\} \cup \{x \in D : f(x) < -\sqrt{\alpha}\}, & \alpha \geq 0 \end{cases}$$

$\square$

Now $(f+g)^2 = f^2 + 2fg + g^2$ so

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2} \in \mathcal{L}(D)$$

$(f, g, f+g, f^2, g^2, \frac{1}{2}(f+g)^2, -\frac{1}{2}f^2, -\frac{1}{2}g^2 \in \mathcal{L}(D))$

54] Let $f_1, f_2 \dots \in \mathcal{S}(D)$. Then

i) $\sup\{f_1, \dots, f_n\} \in \mathcal{S}(D)$

$(h(x) = \sup(f_1(x), \dots, f_n(x)))$

ii) $\sup\{f_1, f_2 \dots\} \in \mathcal{S}(D)$.

iii) $\overline{\lim_{n\to\infty}} f_n \in \mathcal{S}(D)$

iv) $\inf\{f_1, \dots, f_n\} \in \mathcal{S}(D)$.

v) $\inf\{f_1, f_2 \dots\} \in \mathcal{S}(D)$

vi) $\underline{\lim}_{n\to\infty} f_n \in \mathcal{S}(D)$.

≥ 0 works

Proof i) know $\{x \in D : \sup\{f_1(x), \dots, f_n(x)\} < a\}$

$$= \bigcap_{i=1}^{n} \underbrace{\{x \in D : f_i(x) < a\}}_{\in \mathfrak{F}_M} \in \mathfrak{F}_M$$

ii) $\{x \in D : \sup(f_1(x), f_2(x), \dots) < a\} =$

$$\bigcap_{i=1}^{\infty} \underbrace{\{x \in D : f_i(x) < a\}}_{\in \mathfrak{F}_M} \in \mathfrak{F}_M$$

iv) $\{x \in D : \inf(f_1(x), \dots, f_n(x)) > a\} =$

$$\bigcap_{i=1}^{n} \{x \in D : f_i(x) > a\} \in \mathfrak{F}_M$$

v) $\{x \in D : \inf(f_1(x), f_2(x), \dots) > a\}$

$= \bigcap_{i=1}^{\infty} \{x \in D : f_i(x) > a\} \in \mathfrak{J}_m$

iii) $\overline{\lim_{n\to\infty}} f_n = \inf_k \underbrace{\sup_{m \geq k} f_m}_{g_k} = \inf_k g_k \in \mathfrak{J}(D)$

by v).

vi) $\underline{\lim_{n\to\infty}} f_n = \sup_k \underbrace{\inf_{m \geq k} f_m}_{g_k} = \sup_k g_k \in \mathfrak{J}(D)$

by iii).

$\square$

55) Th) If $f_n \in \mathfrak{J}(D)$ and $\lim_{n\to\infty} f_n = f$,

then $f \in \mathfrak{J}(D)$.

proof) know $f = \underline{\lim_{n\to\infty}} f_n = \overline{\lim_{n\to\infty}} f_n \in \mathfrak{J}(D)$

by 54 iii) and vi).

56) <u>know</u> Def) A property is said to hold <u>almost everywhere</u> (ae or a.e.)

if the set of points where the

property fails to hold is a set of measure zero.

ex] Let $f(x) = \begin{cases} 5, & x \in \mathbb{Q} \\ 2, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$

Since $m(\mathbb{Q})=0$, $f(x)=2$ ae

57] Th Let $f, g: D \to Y_q$, $f \in \mathcal{J}(D)$, and $f(x)=g(x)$ ae. Then $g \in \mathcal{J}(D)$.

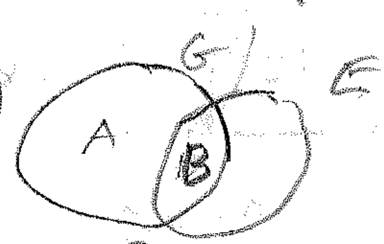
proof] Let $a \in \mathbb{R}$. ~~Let~~ $E = \{x \in D : f(x) \neq g(x)\}$. $m(E)=0$.

Let $F = \{x \in D : f(x) < a\} \in \mathfrak{F}_m$.

Let $G = \{x \in D : g(x) < a\}$.

Let $B = \{x \in E : g(x) < a\} = G \cap E$. $m(B)=0$

Thus $G = \underbrace{(G \cap E^c)}_{A} \cup (G \cap E)$



$= \underbrace{\{x \in D\setminus E : f(x) < a\}}_{A} \cup \{x \in E : g(x) < a\}$ — $g(x) \neq f(x)$ on $B \subseteq E$

$g(x)=f(x)$ on set $A = G \cap E^c \subseteq E^c$.

$= \left(\underbrace{\{x \in D : f(x) < a\}}_{\in \mathfrak{F}_m} \cap \underbrace{E^c}_{\in \mathfrak{F}_m \text{ since } E \in \mathfrak{F}_m}\right) \cup \underbrace{\{x \in E : g(x) < a\}}_{B \subseteq E \text{ so } m(B)=0 \text{ so } B \in \mathfrak{F}_m} \in \mathfrak{F}_m$ $\square$

58} A measurable function can be well approximated by a) a step function b) by a continuous function. 501 47

59} If $f: D \to Y$ is a continuous function, then $f \in \mathcal{B}(D)$.

60} Th Let f be a measurable function with $D = [a,b]$, and assume f takes on values $\pm\infty$ only on a set of measure 0. Then given $\varepsilon > 0$, we can find a step functions g and a continuous function h such that $|f-g| < \varepsilon$ and $|f-h| < \varepsilon$ except on a set of measure $< \varepsilon$

$m\{x : |f(x) - g(x)| \geq \varepsilon\} < \varepsilon$ and

$m\{x : |f(x) - h(x)| \geq \varepsilon\} < \varepsilon$.

§3.6 61} Littlewood's 3 principles ($X = \mathbb{R}$):

a) Every measurable set is nearly a finite union of intervals.

b) Every measurable function is nearly continuous.

c) Every convergent sequence of measurable functions is nearly uniformly convergent.

62) know Egoroff's Theorem If $\{f_n\}$ is a sequence of measurable functions that converge to a real valued function f ae on a measurable set E, then given $\eta > 0$, $\exists$ $A \subseteq E$ with $m(A) < \eta$ such than f_n converges to f uniformly on $E - A$.

63) If $f_1(u)$ is continuous and $u = f_2(x)$ is measurable, then the composite function $h = f_1 \circ f_2 = f_1(f_2(x))$ is measurable.

64) (63) need not hold if f is measurable but not continuous.

65] Thm A function f is continuous sol 48 on $\mathbb{R}$ iff $f^{-1}[O]$ is an open set for any open set $O \in \mathbb{R}$.

proof that $f: \mathbb{R} \to X$ is continuous on $\mathbb{R} \Rightarrow$ f is measurable:

Let $a \in \mathbb{R}$. Then

$$f^{-1}[\underbrace{(-\infty, a)}_{\text{open set}}] = \{x \in \mathbb{R} ;\ f(x) < a\}$$

is an open set $\in \mathcal{F}_M$ by 65].

$\Big(\mathcal{B}(\mathbb{R}) \subseteq \mathcal{F}_M$ so open sets are in $\mathcal{F}_M \Big)$

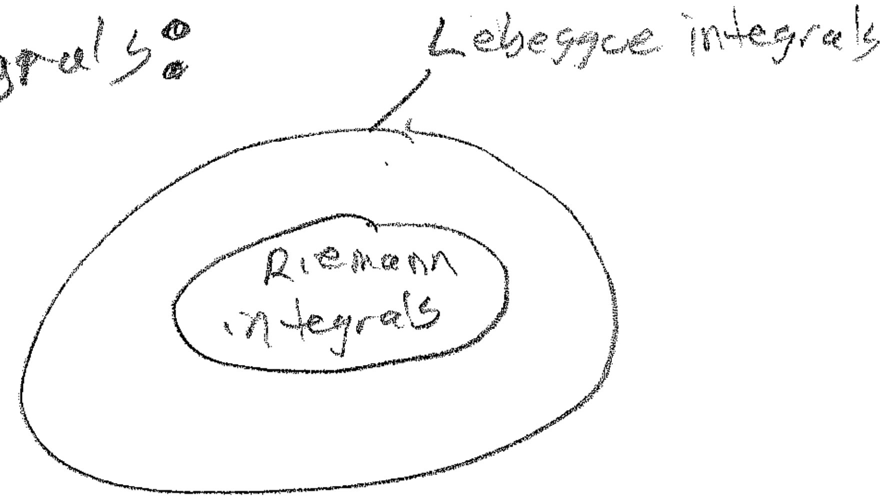
$\square$

Ch 4 Leb. Integral (most important ch for M501)

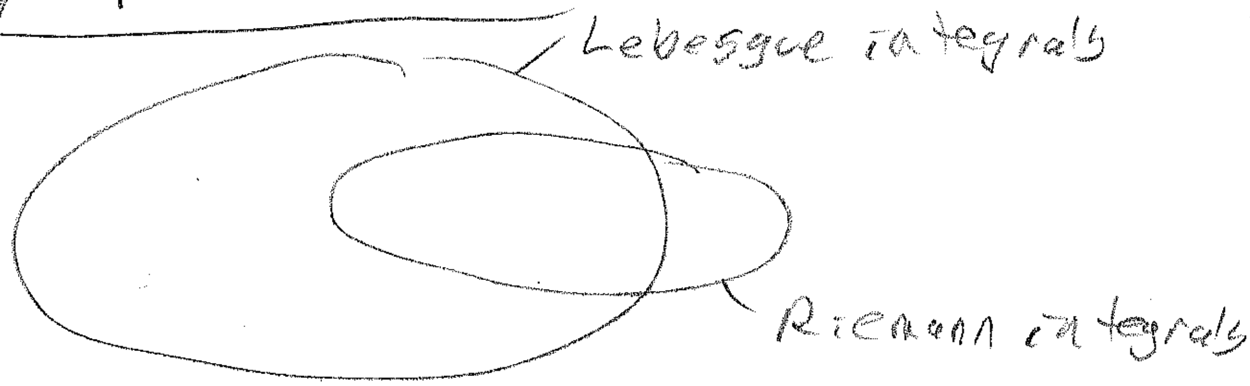
1] Denote $\int_a^b f(x)\,dx$ as the Lebesgue integral and $(R)\int_a^b f(x)\,dx$ as the Riemann integral (calc I). If $(R)\int_a^b f(x)\,dx$ exists, then $(R)\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$ and f(x) is a L measurable function.

2) proper integrals:

f bounded in $[a,b]$



3) <u>improper integrals:</u>



4) Riemann integral: Let f be a bounded function in $[a,b]$. Let partition

$\Delta = \{x_0, x_1, \ldots x_n\}$ with

$a = x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n = b$.

$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

Let $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$

(every bounded function on a closed interval has a sup)

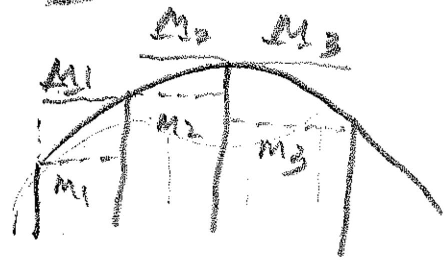
$$m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$$

upper sum $\sum_{i=1}^{n} M_i (x_i - x_{i-1}) = U(\Delta) = U(f, \Delta)$

lower sum $\sum_{i=1}^{n} m_i (x_i - x_{i-1}) = L(\Delta) = L(f, \Delta)$

upper integral $= \overline{\int_a^b} f(x)\,dx = \inf_{\Delta} U(\Delta)$

lower integral $\underline{\int_a^b} f(x)\,dx = \sup_{\Delta} L(\Delta)$



5) Def! Let f be bounded in $[a,b]$.
Then f is <u>Riemann integrable</u> iff

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

6) Set $E_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$,

Then $k_i \chi_{E_i}(x) = \begin{cases} k_i & x \in E_i \\ 0 & x \notin E_i. \end{cases}$

eg $k_i = M_i$ or $k_i = m_i$

Define $\int \chi_{E_i}(x)\,dx = m(E_i) = x_i - x_{i-1}$.
(this will be the L integral if $E_i \in \mathfrak{M}_M$).

The step function $\phi(x) = \sum_{i=1}^{n} \underline{M}_i \chi_{E_i}(x) \geq f(x)$,

$$\psi(x) = \sum_{i=1}^{n} m_i \chi_{E_i}(x) \leq f(x).$$

7} Def: $\int_a^b \phi(x)\,dx = \sum_{i=1}^{n} \underline{M}_i \int \chi_{E_i}(x)\,dx =$

$$\sum_{i=1}^{n} \underline{M}_i \,(x_i - x_{i-1}) = U(\Delta)$$

8} Th $\displaystyle\inf_{\substack{\phi \geq f,\\ \phi \text{ step}}} \int_a^b \phi(x)\,dx = \overline{\int_a^b} f(x)\,dx = \inf_{\Delta} U(\Delta).$

9} Th f is Riemann integrable in $[a,b]$

$$\Leftrightarrow \inf_{\substack{\phi \geq f\\ \phi \text{ step}}} \int_a^b \phi(x)\,dx = \sup_{\substack{\psi \leq f\\ \psi \text{ step}}} \int_a^b \psi(x)\,dx.$$

§4.2 Lebesgue Integral of a Bounded Function f over a set of finite measure

10} A function $\phi(x)$ is a <u>simple function</u> if ϕ is measurable and ϕ only assumes finitely many values. Let the set of nonzero values of simple function ϕ be $\{a_1, \dots, a_n\} = R_0$. (The range of ϕ is either R_0 or $R_0 \cup \{0\}$.)

501 503

Let $A_i = \{x \in \mathbb{R} : \phi(x) = a_i\}$.

Since the a_i are distinct, the A_i are disjoint. Since ϕ is measurable, $A_i \in \mathcal{F}_m$.

11) Def. $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}$ is the <u>canonical representation</u> of ϕ.

12) The canonical representation of ϕ is unique, but can be hard to find.

ex) $\underbrace{\chi_{[0,1]}}_{\text{canonical}} = \underbrace{\chi_{[0,\frac{1}{2})} + \chi_{[\frac{1}{2},1]}}_{\text{not canonical}}$

13) Def) Let ϕ be a simple function vanishing outside a set of finite measure. Let $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ be the canonical representation of ϕ. Then

$$\underbrace{\int \phi(x)\, dx = \int \phi}_{\text{integral}} = \sum_{i=1}^{n} a_i\, m(E_i).$$

14) The Def in 13) is well defined since the canonical representation of ϕ is unique.

15) Know) $\int \chi_E = m(E)$

if χ_E is a simple function so it E is a measurable set.

16) The following lemma removes the restriction of using the canonical rep. of simple function ϕ since we could have $a_i = a_j$ for $i \neq j$.

17) Lemma: Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $E_i \cap E_j = \phi$ for $i \neq j$ and $a_i \neq 0$. Suppose $E_i \in \mathfrak{M}$ and $m(E_i) < \infty$. Then $\int \phi = \sum_{i=1}^{n} a_i\, m(E_i)$.

proof) Let $\{b_1, \ldots, b_m\}$ be the distinct nonzero values of ϕ.

Let $B_i = \{ x \in \mathbb{R} : \phi(x) = b_i \}$.