Then $B_i = \bigcup_{j:a_j=b_i} E_j$ and

$m(B_i) = \sum_{j:a_j=b_i} m(E_j)$. Thus

$\phi(x) = \sum_{i=1}^{M} b_i \chi_{B_i}(x)$ is the canonical representation of ϕ.

$\therefore \int \phi(x)\,dx \stackrel{\text{def}}{=} \sum_{i=1}^{M} b_i\, m(B_i)$

$= \sum_{i=1}^{M} b_i \sum_{j:a_j=b_i} m(E_j) = \sum_{j=1}^{n} a_j\, m(E_j)$ $\square$

write out an example to see this
eg $a_1, a_3 = b_1$; $a_2, a_4, a_5 = b_2$; $a_6, a_7 = b_3$

(8} Th Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then $\int (a\phi + b\psi) = a\int\phi + b\int\psi$

Proof} Let the canonical rep's be

$\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$.

Let $A_0 = \{ x \in \mathbb{R} : \phi(x) = 0 \}$
$B_0 = \{ x \in \mathbb{R} : \psi(x) = 0 \}$ } can. rep. has $a_i \neq 0$, $b_j \neq 0$

Then $\mathbb{R} = \bigcup_{i=0}^{n} A_i = \bigcup_{j=0}^{m} B_j$.

Claim: $a\phi(x) + b\psi(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} (a a_i + b b_j) \chi_{A_i \cap B_j}$

Pf: Note that $\left(\bigcup_{i=0}^{n} A_i \right) \cap \left(\bigcup_{j=0}^{m} B_j \right)$

$= \bigcup_{i=0}^{n} \bigcup_{j=0}^{m} (A_i \cap B_j)$.

The $A_i \cap B_j$ are disjoint:

$(A_{i_0} \cap B_{j_0}) \cap (A_i \cap B_j) = \emptyset$, $(i_0, j_0) \neq (i, j)$

(By the can. rep's, $A_i \cap A_{i_0} = \emptyset$ for $i \neq i_0$ and $B_j \cap B_{j_0} = \emptyset$ for $j \neq j_0$.)

Hence for $x_0 \in \mathbb{R}$ $\exists$ unique i_0, j_0 $\ni x_0 \in A_{i_0} \cap B_{j_0}$. Thus

$\sum_{i=0}^{n} \sum_{j=0}^{m} \underbrace{(a a_i + b b_j) \chi_{A_i \cap B_j}(x_0)}_{0 \text{ except for } (i_0, j_0)} = a a_{i_0} + b b_{j_0}$

$= a\phi(x_0) + b\psi(x_0)$ (only $\chi_{A_{i_0}}(x_0)$ and $\chi_{B_{j_0}}(x_0)$ are nonzero)

$\square$ claim proved

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Now $\int \left[\sum_{i=0}^{n} \sum_{j=0}^{m} (a a_i + b b_j) \chi_{A_i \cap B_j} \right]$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} (a a_i + b b_j) m(A_i \cap B_j)$$

by 17} $= \sum_{i=0}^{n} \sum_{j=0}^{m} a a_i m(A_i \cap B_j) + \sum_{i=0}^{n} \sum_{j=0}^{m} b b_j m(A_i \cap B_j)$

$$= a \sum_{i=0}^{n} a_i \sum_{j=0}^{m} m(A_i \cap B_j) + b \sum_{j=0}^{m} b_j \sum_{i=0}^{n} m(A_i \cap B_j)$$

$$= a \sum_{i=0}^{n} a_i m(A_i \cap \underbrace{\bigcup_{j=0}^{m} B_j}_{\mathbb{R}}) + b \sum_{j=0}^{m} b_j m(\underbrace{\bigcup_{i=0}^{n} A_i}_{\mathbb{R}} \cap B_j)$$

disjoint

$$= a \sum_{i=0}^{n} a_i m(A_i) + b \sum_{j=0}^{m} b_j m(B_j) = a \int \phi + b \int \psi.$$

$\square$

Next, suppose the E_i are not disjoint.

19} Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ vanish outside a set of finite measure.

Then $\int \phi = \sum_{i=1}^{n} a_i m(E_i)$. Linearity for step functions

proof} By 18} and induction, if ϕ_i are simple functions that vanish outside a set of finite measure,

then $\int \sum_{i=1}^{n} a_i \phi_i = \sum_{i=1}^{n} a_i \int \phi_i$.

Take $\phi_i = \chi_{E_i}$ (which is canonical).

then $\int \phi = \sum_{i=1}^{n} a_i \int \chi_{E_i} = \sum_{i=1}^{n} a_i\, m(E_i)$.

$\square$

20} * Def} $\int_E \phi = \underbrace{\int \phi \chi_E}$, $E \in \mathfrak{M}$.

Here $\int \phi = \int_{\mathbb{R}} \phi$, $\int_{\mathbb{R}} \phi(x)\chi_E(x)dx$, and ϕ is a simple function that vanishes outside a set of finite measure.

21} If $A, B \in \mathfrak{M}$, $A \cap B = \emptyset$ and ϕ is defined as in 19} (ϕ vanishes outside of a set of finite measure),

then $\int_{A \cup B} \phi = \int_A \phi + \int_B \phi$. see HW 7

$\chi_{A \cup B} = \chi_A + \chi_B$ if $A \cap B = \emptyset$

22} Let ϕ and ψ be as in 19}.

If $\phi \geq \psi$ a.e., then $\int \phi \geq \int \psi$.

see HW 7.

monotonicity for simple functions

23] P.79 Prop 4.3] Let f be defined 501 53
and bounded on a measurable set E with
$m(E) < \infty$. Then $f \in \mathcal{R}(E)$ iff

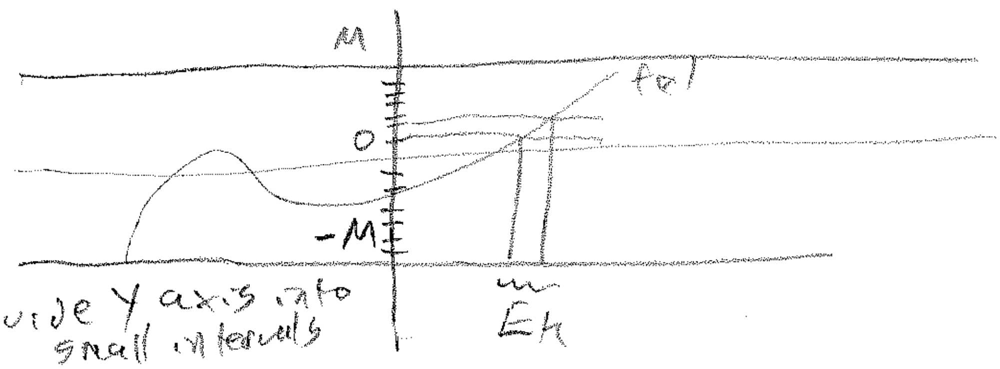
$$\inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi(x)\,dx = \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_E \phi(x)\,dx = A$$

proof] $(\Rightarrow)$ Since f is bounded on E, $\exists\, M > 0$
$\ni |f(x)| \leq M$ for all $x \in E$.
trick For any $n > 0$, divide $[-M, M]$ into
$2n$ subintervals $[-M, -M + \frac{M}{n}), [-M + \frac{M}{n}, -M + 2\frac{M}{n}),$
$\ldots \quad [-M + k\frac{M}{n}, -M + (k+1)\frac{M}{n}), \ldots$

$[-M + (2n-2)\frac{M}{n}, -M + (2n-1)\frac{M}{n}), [-M + (2n-1)\frac{M}{n}, M].$

trick Let $E_k = f^{-1}\left[[-M + \frac{Mk}{n}, -M + \frac{M}{n}(k+1))\right]$

$= \left\{x \in E : -M + \frac{Mk}{n} \leq f(x) < -M + \frac{M}{n}(k+1)\right\}$
(except E_{2n-1} is closed) $k = 0, \ldots, 2n-1$.



divide Y axis into small intervals

Let $\psi_n(x) = \sum_{k=0}^{2n-1} \left(-M + \frac{M}{n}(k+1)\right) \chi_{E_k}(x) \geq f(x)$

$$\phi_n(x) = \sum_{k=0}^{2n-1} \left(-M + \frac{M}{n} k\right) \chi_{E_k}(x) \leq f(x)$$

(draw your own picture)

Then $0 \leq \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi - \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_E \phi$ (*)

$$\leq \int_E \psi_n - \int_E \phi_n =$$

$$\sum_{k=0}^{2n-1} \left(-M + \frac{M}{n}(k+1)\right) m(E_k) -$$

$$- \sum_{k=0}^{2n-1} \left(-M + \frac{M}{n} k\right) m(E_k) =$$

$$\sum_{k=0}^{2n-1} \frac{M}{n} m(E_k) = \frac{M}{n} \sum_{k=0}^{2n-1} m(E_k) = \underbrace{\frac{M}{n} m(E)}_{\to 0 \text{ as } n \to \infty}.$$

Thus (*) $= 0$. and A holds.

Note! $E_k \subseteq E$, the E_k are disjoint, $\bigcup_k E_k = E$.

y axis has $\left[-M + \frac{M}{n} k, -M + \frac{M}{n}(k+1)\right)$.

x axis has $E_k = f^{-1}\left[\left[-M + \frac{M}{n} k, -M + \frac{M}{n}(k+1)\right)\right]$

(could be a union of disjoint intervals)

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($\Leftarrow$) By def of sup, $\forall \varepsilon > 0$, $\exists x_0 \ni$ $x_0 > A - \varepsilon \iff 0 < A - x_0 < \varepsilon$.

suppose A holds.

For each n, by def of inf, $\exists\, \psi_n(x) \geq f$

$\ni\ 0 < \int_E \psi_n - A < \frac{1}{2n}$.

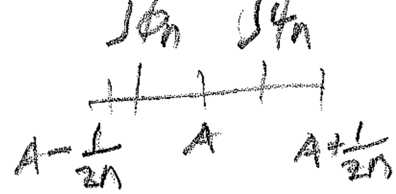
By def of sup, for each n, let $\varepsilon = \frac{1}{2n}$.

Then $\exists\, \phi_n \leq f \ni 0 < A - \int_E \phi_n \leq \frac{1}{2n}$

So for each n, $\exists\, \phi_n \leq f \leq \psi_n \ni$

$$\int_E \psi_n - \int_E \phi_n \leq 2\frac{1}{2n} = \frac{1}{n}.$$

Let $\psi_* = \inf_n \psi_n$ and $\phi_* = \sup_n \phi_n$.



Since $f \leq \psi_n\ \forall n$, f is a lower bound of ψ_n.

$\therefore\ f \leq \inf_n \psi_n = \psi_*$

Since $f \geq \phi_n\ \forall n$, f is an upper bound of ϕ_n.

$\therefore\ f \geq \sup_n \phi_n = \phi_*$.

Also, ψ_* and ϕ_* are measurable functions (sup and inf of measurable functions are measurable functions).

claim: $f = \phi_*$ a.e. in E. $f \geq \phi_*$

pf} Since $\{x \in E : f \neq \phi_*\} =$

$\{x \in E : \phi_*(x) < f(x)\} = C$

$\subseteq \{x \in E : \phi_*(x) < \psi_*(x)\} = B,$

gets rid of f

if $m(B) = 0$ then $m(C) = 0$

and $f = \phi_*$ a.e. $\therefore f$ is measurable.

May have you show $m(B) = 0$ on HW.

$\square$

24) know Def} Let f be a bounded measurable function on E with $m(E) < \infty$. Then define the L. integral

$$\int_E f(x)\,dx = \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_E \psi(x)\,dx = \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_E \phi(x)\,dx.$$

prop 4.3 tells $f \in \mathcal{R}(E)$ so "measurable" is redundant

25} i) If $E = [a,b]$,

$$\int_E f(x)\,dx = \int_E f = \int_a^b f(x)\,dx = \int_a^b f.$$

ii) $\int_E f = \int f \chi_E = \int_E f(x)\,dx$.

26} Prop 4.5 Suppose f and g are bounded measurable functions on E with $m(E) < \infty$.

i) linearity: $\int_E (af+bg) = a\int_E f + b\int_E g$

ii) If $f = g$ a.e. on E, then $\int_E f = \int_E g$.

iii) monotonicity; If $f \le g$ a.e. on E, then $\int_E f \le \int_E g$

Thus $\left|\int_E f\right| \le \int_E |f|$.

iv) If $a \le f(x) \le b$ on E, then $a\,m(E) \le \int_E f \le b\,m(E)$

(use $\int \chi_E = m(E)$).

v) If $A \cap B = \emptyset$, $A \cup B = E$, A, B measurable, then $\int_{A\cup B} f = \int_A f + \int_B f$.

Proof of v) $\int_{A\cup B} f = \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_{A\cup B} \psi$

$$= \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \left(\int_A \psi + \int_B \psi \right)$$

Simple functions break this way over disjoint sets by HW7

$$\geq \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_A \psi + \inf_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_B \psi$$

$$\geq \int_A f + \int_B f.$$

use inf to get LHS ≥ RHS and use sup to get RHS ≥ LHS

Now $\int_{A\cup B} f = \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_{A\cup B} \phi$

$$= \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \left(\int_A \phi + \int_B \phi \right)$$

$$\leq \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_A \phi + \sup_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_B \phi = \int_A f + \int_B f$$

Thus $\int_A f + \int_B f \leq \int_{A\cup B} f \leq \int_A f + \int_B f$.

$\square$

27) Th. Let f be a bounded function on $[a,b]$. If f is Riemann integrable on $[a,b]$ then f is L. integrable on $[a,b]$.
(Hence f is L. measurable on $[a,b]$.)

proof) Want to use prop 4.3. So want to prove that
$$\sup_{\substack{\phi \le f \\ \phi \text{ simple}}} \int_{[a,b]} \phi = \inf_{\substack{\psi \ge f \\ \psi \text{ simple}}} \int_{[a,b]} \psi.$$

It is enough to show LHS $\ge$ RHS
Since RHS $\ge$ LHS.

Now
$$(R)\int_a^b f(x)\,dx = \sup_{\substack{\phi \le f \\ \phi \text{ step}}} \int_a^b \phi = \inf_{\substack{\psi \ge f \\ \psi \text{ step}}} \int_a^b \psi$$

Since step functions are simple functions,
$$\inf_{\substack{\psi \ge f \\ \psi \text{ simple}}} \int_a^b \psi \le \inf_{\substack{\psi \ge f \\ \psi \text{ step}}} \int_a^b \psi = \sup_{\substack{\phi \le f \\ \phi \text{ step}}} \int_a^b \phi \le \sup_{\substack{\phi \le f \\ \phi \text{ simple}}} \int_a^b \phi$$

Since $\inf\limits_{\substack{\psi \geq f \\ \psi \text{ simple}}} \int_a^b \psi = \sup\limits_{\substack{\phi \leq f \\ \phi \text{ simple}}} \int_a^b \phi$,

we get equality, $\therefore$ f is L. integrable

(and thus L. measurable by prop 4.3)

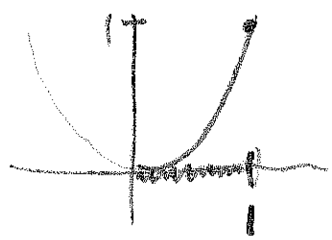
$\square$

28) Prop 4.7} A bounded function f on $[a,b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero. (So f is contin a.e. on $[a,b]$.)

ex] $f(x) = \begin{cases} x^2 & x \notin \mathbb{Q} \cap [0,1] \\ 0 & x \in \mathbb{Q} \cap [0,1] \end{cases}$

i) Does (R) $\int_0^1 f(x)\,dx$ exist?

ii) Does (L) $\int_0^1 f(x)\,dx$ exist?



Soln] For $x_0 = 0$ f is continuous

$x_0 \neq 0$ f is discontinuous.

Hence f is continuous at 0 but f is not continuous for $x \in (0,1]$.

$m(\{x \in [0,1] : f \text{ is not continuous}\})$

$= m([0,1]) = 1 > 0.$

$\therefore$ f is not Riemann integrable.

ii) Set $g(x) = x^2$. Then $g = f$ ae on $[0,1]$. $\therefore$ f is L. integrable

$$(L)\int_0^1 f(x)\,dx = (R)\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3.$$

$\int_0^1 f = \int_0^1 g$

end exam 2 material

29) We can't, in general, exchange the limit and integral operators.

For ex, let $f_n(x) = \begin{cases} n & x \in [0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1]. \end{cases}$

Then $\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in (0,1]$.

$\therefore$ $\lim_{n\to\infty} f_n(x) = 0 = f$ ae on $[0,1]$.

$\int_0^1 f = 0$, $\int_0^1 f_n = 1$

$\therefore$ $\lim_{n\to\infty} \int_0^1 f_n = 1 \neq \int_0^1 \lim_{n\to\infty} f_n = \int 0 = 0.$

30] There will be several important exceptions that hold for L. integrals but not for Riemann integrals.

31] Know Bounded Convergence Theorem

(BCT): Let $E \in \mathfrak{M}$, $m(E) < \infty$ and $\underbrace{f_n \in \mathcal{S}(E)}_{\text{measurable}}$. If $\exists$ $M > 0$ $\ni$

$|f_n(x)| \leq M$ $\forall n \in \mathbb{N}$ and $\forall x \in E$,

and if $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in E$, (so $f \in \mathcal{S}(E)$)

then $\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$.

proof] $\lim_{n\to\infty} f_n = f \in \mathcal{S}(E)$ since $f_n \in \mathcal{S}_E$.

Since $|f_n| \leq M$ $\forall x$ and n, $|f(x)| \leq M$ $\forall x \in E$.

So f is bounded and measurable on E, and $\int_E f$ exists.

Proving $\lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} a_n = a = \int_E f$

is equivalent to showing that