Math 580 HW 2 Spring 2022. Due Wednesday, Jan. 26.
Numbers in parentheses such as (1.7) refer to the problem in the text.

SIX PROBLEMS: Get TWO MORE PROBLEMS ON THE NEXT PAGE

Recall integrals by u-substitution. $I = \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = \int_c^d f(u)du = F(u)|_c^d = F(d) - F(c) = F(u)|_{g(a)}^{g(b)} = F(g(x))|_a^b = F(g(b)) - F(g(a))$ where $F'(x) = f(x)$, $u = g(x)$, $du = g'(x)dx$, $d = g(b)$, and $c = g(a)$.

1. (1.7). The following problem uses the Gamma function and u-substitution to show that the normal density integrates to 1 (usually shown with polar coordinates). When you perform the u-substitution, make sure you say what $u = g(x)$, $du = g'(x)dx$, $d = g(b)$, and $c = g(a)$ are.

a) Let $f(x)$ be the pdf of a $N(\mu, \sigma^2)$ random variable. Perform u-substitution on $I = \int_{-\infty}^{\infty} f(x)dx$ with $u = (x - \mu)/\sigma$.

b) Break the result into two parts, $I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2} du + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-u^2/2} du$. Then perform u-substitution on the first integral with $v = -u$.

c) Since the two integrals are now equal, $I = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-v^2/2} dv = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-v^2/2} \frac{1}{v} v dv$. Perform u-substitution with $w = v^2/2$.

d) Using the Gamma function (p. 292) show that $I = \Gamma(1/2)/\sqrt{\pi} = 1$.

2. (1.8). Let X be a $N(0, 1)$ (standard normal) random variable. Use integration by parts to show that $EX^2 = 1$. Recall that integration by parts is used to evaluate $\int f(x)g'(x)dx = \int u dv = uv - \int v du$ where $u = f(x)$, $dv = g'(x)dx$, $du = f'(x)dx$ and $v = g(x)$. When you do the integration, clearly state what these 4 terms are (eg $u = x$).

3. (1.12). Suppose the random variable X has cdf $F_X(x) = 0.9\ \Phi(x-10) + 0.1\ F_W(x)$ where $\Phi(x-10)$ is the cdf of a normal $N(10, 1)$ random variable with mean 10 and variance 1 and $F_W(x)$ is the cdf of the random variable W that satisfies $P(W = 200) = 1$.
a) Find $E\ W$.
b) Find $E\ X$.

4. (1.26). Suppose that X has pdf

$$f(x) = \frac{h(x)e^{\theta x}}{\lambda(\theta)}$$

for $x \in \mathcal{X}$ and for $-\infty < \theta < \infty$ where $\lambda(\theta)$ is some positive function of θ and $h(x)$ is some nonnegative function of x. Find the moment generating function of X using the kernel method. Your final answer should be written in terms of λ, θ and t.

5. (2.74). Suppose that the joint probability mass function $f(y_1, y_2)$ of Y_1 and Y_2 is given in the following table.

$f(y_1, y_2)$		y_2 = 0	1	2
y_1	0	0.38	0.14	0.24
	1	0.17	0.02	0.05

a) Find the marginal probability function $f_{Y_2}(y_2)$ for Y_2.

b) Find the conditional probability function $f(y_1|y_2)$ of Y_1 given $Y_2 = 2$.

6. Let $X \sim Poisson(\theta)$, $Y \sim Poisson(\lambda)$, and $Z = X + Y \sim Poisson(\theta + \lambda)$ where X and Y are independent. Show that $X|Z = z$ has a binomial distribution with $\rho = \theta/(\theta + \lambda)$. (See hint given in class.)