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Math 580    HW 9 Spring 2022.  Due Wednesday, April 6.
Quiz 8 is on MLEs, sufficient, minimal sufficient and complete
sufficient statistics.  LONG HOMEWORK: 2 pages, 7 problems
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1) (5.4) Let $X_1, ..., X_n$ be iid $N(\mu, \gamma_o^2 \mu^2)$ random variables where $\gamma_o^2 > 0$ is **known** and $\mu > 0$. Find the log likelihood $\log(\text{L}(\mu|X_1, ..., X_n))$ and solve

$$\frac{d}{d\mu} \log(L(\mu|X_1, ..., X_n)) = 0$$

for $\hat{\mu}_o$, a potential candidate for the MLE of μ. (Hint: using $L(\mu) = f(\boldsymbol{x}|\mu)$ where $f(\boldsymbol{x}|\mu)$ was found in HW7 problem 3a may be useful.)

2) (6.1) Let W be an estimator of $\tau(\theta)$. Show that

$$MSE_{\tau(\theta)}(W) = Var_\theta(W) + [Bias_{\tau(\theta)}(W)]^2.$$

3) (6.2) Let $X_1, ..., X_n$ be independent identically distributed random variable from a $N(\mu, \sigma^2)$ distribution. Hence $E(X_1) = \mu$ and $VAR(X_1) = \sigma^2$. Consider estimates of σ^2 of the form

$$S^2(k) = \frac{1}{k} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

where $k > 0$ is a constant to be chosen. Determine the value of k which gives the smallest mean square error. (Hint: Find the MSE as a function of k, then take derivatives with respect to k. Also, use Theorem 4.1c and Remark 5.1 VII.)

4) (6.3) Let $X_1, ..., X_n$ be iid $\text{N}(\mu, 1)$ random variables. Find $\tau(\mu)$ such that $T(X_1, ..., X_n) = (\sum_{i=1}^n X_i)^2$ is the UMVUE of $\tau(\mu)$.

5) (6.4) Let $X \sim N(\mu, \sigma^2)$ where σ^2 is known. Find the Fisher information $I_1(\mu)$.

6) (6.5) Let $X \sim N(\mu, \sigma^2)$ where μ is known. Find the Fisher information $I_1(\sigma^2)$.

Next page has problem 7).

7) (6.6) Let $X_1, ..., X_n$ be iid $N(\mu, \sigma^2)$ random variables where μ is **known** and $\sigma^2 > 0$. Then $W = \sum_{i=1}^n (X_i - \mu)^2$ is a complete sufficient statistic and $W \sim \sigma^2 \chi^2_n$. From Chapter 10,

$$EY^k = \frac{2^k \Gamma(k + n/2)}{\Gamma(n/2)}$$

if $Y \sim \chi^2_n$. Hence

$$T_k(X_1, ..., X_n) \equiv \frac{\Gamma(n/2) W^k}{2^k \Gamma(k + n/2)}$$

is the UMVUE of $\tau_k(\sigma^2) = \sigma^{2k}$ for $k > 0$. Note that $\tau_k(\theta) = (\theta)^k$ and $\theta = \sigma^2$.

a) Show that

$$Var_\theta T_k(X_1, ..., X_n) = \sigma^{4k} \left[\frac{\Gamma(n/2)\Gamma(2k + n/2)}{\Gamma(k + n/2)\Gamma(k + n/2)} - 1 \right] \equiv c_k \sigma^{4k}.$$

b) Let $k = 2$ and show that $\text{Var}_\theta[T_2] - FCRLB(\tau_2(\theta)) > 0$ where FCRLB($\tau_2(\theta)$) is for estimating $\tau_2(\sigma^2) = \sigma^4$ and $\theta = \sigma^2$.

(CRLB = FCRLB)