bin(0,p) is the point mass at 0

likely $Y \sim$ Pois(λp)

Math 580   2017   Exam 1   Name___________

1) Suppose that the conditional distribution of $Y|X$ is a binomial(X, p) distribution and that the random variable X has a Poisson(λ) distribution.

e a) Find E(Y). $= E(E(Y|X)) = E(Xp) = pE(X) = \boxed{p\lambda = \lambda p}$

480 Q4d70 E3d18

b) Find Var(Y). $= V(E(Y|X)) + E(V(Y|X))$

$= V(Xp) + E[X\,p(1-p)]$

$= p^2 V(X) + p(1-p)E(X) = \boxed{p^2\lambda + p(1-p)\lambda = p\lambda = \lambda p}$

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2) Let X have pdf

$$f(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-x/\theta}}{x\Gamma(\alpha)}$$

5/9 got it

M403 ← gamma(α, $\theta = \lambda$) pdf but θ^α is not in the denom

for where $x, \alpha, \theta > 0$. Find $E(X^k)$ using the kernel method.

$$EX^k = \int_0^\infty \frac{x^k \left(\frac{x}{\theta}\right)^\alpha e^{-x/\theta}}{x\,\Gamma(\alpha)}\,dx = \int_0^\infty \frac{x^k \frac{\theta^k}{\theta^k}\left(\frac{x}{\theta}\right)^\alpha e^{-x/\theta}}{x\,\Gamma(\alpha)}\,dx$$

$$= \frac{\theta^k\,\Gamma(\alpha+k)}{\Gamma(\alpha)} \underbrace{\int_0^\infty \frac{\left(\frac{x}{\theta}\right)^{\alpha+k} e^{-x/\theta}}{x\,\Gamma(\alpha+k)}\,dx}_{1 = \int_0^\infty f(x|\theta, \alpha+k)\,dx \text{ pdf}}$$

$$= \boxed{\frac{\theta^k\,\Gamma(\alpha+k)}{\Gamma(\alpha)}}$$

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$Y=4X^2+3$ 19 7 3 7 19

x	−2	−1	0	1	2
f(x)	0.1	0.3	0.3	0.2	0.1

e 3) Let the discrete random variable X have a probability mass function given by the table above. Find the pmf of $Y = 4X^2 + 3$.

y	3	7	19
$f(y)=P(Y=y)$	0.3	0.5	0.2

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e 4) Suppose the the joint pmf $f(y_1, y_2)$ of Y_1 and Y_2 is tabled as shown.

$f(y_1,y_2)$		y_2 = 0	1	2	$f_{Y_1}(y_1)=P(Y_1=y_1)$
y_1	0	0/15	3/15	3/15	6/15
	1	2/15	6/15	0/15	8/15
	2	1/15	0/15	0/15	1/15
$P(Y_2=y_2)=f_{Y_2}(y_2)$		3/15	9/15	3/15	

a) Are Y_1 and Y_2 independent? Explain briefly.

no, support is not a cross product

(hard way! $f(0,0)=0 \neq f_{Y_1}(0)\,f_{Y_2}(0) = \frac{6}{15}\frac{3}{15} > 0$)

b) Find the moment generating function mgf of Y_1.

$\sum e^{ty} f_{Y_1}(y) = e^{t0}\frac{6}{15} + e^{t1}\frac{8}{15} + e^{t2}\frac{1}{15}$

$m(t) = \boxed{\frac{6}{15} + \frac{8}{15}e^{t} + \frac{1}{15}e^{2t}}$

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c) Find $E(Y_1)$ using the mgf from b).

$m'(t) = \frac{8}{15}e^{t} + \frac{1}{15}e^{2t}(2)$

$m'(0) = E(Y_1) = \frac{8}{15} + \frac{2}{15} = \boxed{\frac{10}{15} = \frac{2}{3}}$

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5) Suppose the random variable X has cdf $F_X(x) = 0.7\ F_Z(x) + 0.3\ F_W(x)$ where F_Z is the cdf of a uniform (6,7) random variable and $F_W(x)$ is the cdf of a chi-square(10) random variable.

a) Find $E(X)$. $= 0.7\,E(Z) + 0.3\,E(W) =$

$0.7\ \frac{6+7}{2} + 0.3(10) = \boxed{7.55}$

b) Find $E(X^2)$. $= 0.7\,E(Z^2) + 0.3\,E(W^2)$

$= 0.7\,[V(Z) + (E(Z))^2] + 0.3\,[V(W) + (E(W))^2] =$

$0.7\,\underbrace{\left[\frac{1}{12} + (6.5)^2\right]}_{42\frac{1}{3}} + 0.3\,(20 + 100)$

$= 29.6333 + 36 = \boxed{65.6333}$

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6) Suppose that X is a random variable with pdf

4/9 got 8/8

$$f(x) = \frac{1}{\lambda} x^{\frac{1}{\lambda} - 1},$$

where $0 < x \le 1$ and $\lambda > 0$. Let $Y = -\log(X)$ and find the pdf of Y.

$t(0) = -\log(0) = \infty,\ t(1) = -\log(1) = 0$ so $y > 0$.

$-Y = \log(X),\quad X = e^{-Y} = t^{-1}(Y)$

$\left|\frac{d\,t^{-1}(y)}{dy}\right| = |-e^{-y}| = e^{-y}$

$f_Y(y) = f_X(t^{-1}(y))\left|\frac{d\,t^{-1}(y)}{dy}\right| = \frac{1}{\lambda}(e^{-y})^{\frac{1}{\lambda}-1} e^{-y}$

$= \boxed{\frac{1}{\lambda} e^{-y/\lambda},\quad y > 0}$

or $\ge$

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Spiegel P98

7) Suppose that X_1 and X_2 have joint pdf

$$f_{X_1,X_2}(x_1,x_2) = \frac{x_1\, x_2}{96}$$

for $0 < x_1 < 4$ and $1 < x_2 < 5$. Consider the transformation $Y_1 = X_1 + 2X_2$ and $Y_2 = X_1$.

a) Are X_1 and X_2 independent? Explain briefly.

Yes, $f_{X_1,X_2}(x_1,x_2) = \frac{x_1}{96}(x_2) = h_1(x_1)\, h_2(x_2)$ on $\underbrace{\text{cross product support}}_{\text{or}}$

b) Find the Jacobian J for the transformation.

$X_1 = Y_2 = t_1^{-1}(y_1, y_2)$, $2X_2 = Y_1 - X_1 = Y_1 - Y_2$, or $X_2 = \frac{Y_1 - Y_2}{2} = t_2^{-1}(y)$

$\frac{\partial t_1^{-1}(y)}{\partial y_1} = 0$ $\quad \frac{\partial t_1^{-1}(y)}{\partial y_2} = 1$

$\frac{\partial t_2^{-1}(y)}{\partial y_1} = \frac{1}{2}$ $\quad \frac{\partial t_2(y)}{\partial y_2} = -\frac{1}{2}$

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \boxed{-\frac{1}{2}}$$

c) Find the joint pdf $f(y_1, y_2)$ of Y_1 and Y_2. Include the support. $=$

$$f_{X_1,X_2}\left(y_2, \frac{y_1-y_2}{2}\right)|J| = \frac{y_2 \frac{y_1-y_2}{2}}{96}\,\frac{1}{2}\; \underbrace{I(0<y_2<4)\, I\left(1<\frac{y_1-y_2}{2}<5\right)}_{I(0<y_2<4)\; I(2<y_1-y_2<10)}$$

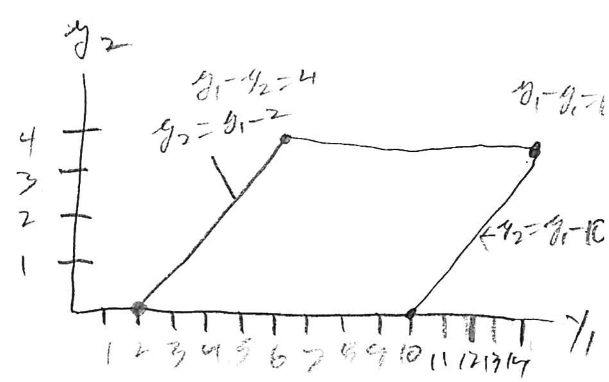
$$= \boxed{\frac{y_2(y_1-y_2)}{384}\, I(2<y_1-y_2<10)\; I(0<y_2<4)}$$

so $2+y_2 < y_1 < 10+y_2$ $\quad 0 < y_2 < 4$

$\leftarrow \frac{5}{5}$ got it

d) Are Y_1 and Y_2 independent? Explain briefly.

No, support is not a cross product or $f_{Y_1,Y_2}(y_1,y_2) \neq h_1(y_1)\, h_2(y_2)$



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