Math 580 2017 Exam 2 Name___________

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1) Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_4\left(\begin{pmatrix} 49 \\ 25 \\ 9 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 3 & 0 \\ -1 & 5 & -3 & 0 \\ 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}\right).$$

a) Find the distribution of X_2. $\sim N(25, 5)$

b) Find the distribution of $(X_1, X_3)^T$. $\sim N_2\left[\begin{pmatrix} 49 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\right]$

c) Which pairs of random variables X_i and X_j are independent?

$X_1 \perp\!\!\!\perp X_4$, $X_2 \perp\!\!\!\perp X_4$, $X_3 \perp\!\!\!\perp X_4$

d) Find the correlation $\rho(X_1, X_3)$. $= \dfrac{\text{cov}(X_1, X_3)}{\sqrt{V(X_1)\ V(X_3)}}$

$$= \frac{3}{\sqrt{2}\sqrt{5}} = \boxed{\frac{3}{\sqrt{10}} = 0.9487}$$

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2) Recall that if $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, then the conditional distribution of $\boldsymbol{X}_1$ given that $\boldsymbol{X}_2 = \boldsymbol{x}_2$ is multivariate normal with mean $\boldsymbol{\mu}_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$ and covariance $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.

Let Y and X follow a bivariate normal distribution

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_2\left(\begin{pmatrix} 49 \\ 17 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -1 & 4 \end{pmatrix} \right).$$

a) Find $E(Y|X)$. $= 49 + (-1)\frac{1}{4}(X-17) = 49 + \frac{17}{4} - \frac{1}{4}X$

$= \boxed{53.25 - 0.25X}$

x_2 fine

b) Find $\mathrm{Var}(Y|X)$.

$= 3 - (-1)\frac{1}{4}(-1) = 3 - \frac{1}{4} = \boxed{2.75}$

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e 3) Suppose $Y_1, ..., Y_n$ are iid with moment generating function

$$m_Y(t) = \exp\left[\frac{\lambda}{\theta}\left(1 - \sqrt{1 - \frac{2\theta^2 t}{\lambda}}\right)\right]$$

for $t < \lambda/(2\theta^2)$ where $\theta > 0$ and $\lambda > 0$. Find the mgf of $W = \sum_{i=1}^n Y_i$.

$= [m_Y(t)]^n = \exp\left[\frac{\lambda n}{\theta}\left(1 - \sqrt{1 - \frac{2\theta^2 t}{\lambda}}\right)\right]$

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$F(x) = 1 - e^{-x} \quad x > 0, \quad f(x) = e^{-x}, \quad x > 0$

4) Let $X_{(1)} = \min_{1 \le i \le n} X_i$. If $X_1, \ldots, X_n$ are iid exponential(1) random variables, find $E(X_{(1)})$.

$$f_{X_{(1)}}(x) = n\left[1 - F(x)\right]^{n-1} f(x) = n\left(e^{-x}\right)^{n-1} e^{-x} = n e^{-nx}, \quad x >$$

$X_{(1)} \sim Exp(\frac{1}{n})$

So $E[X_{(1)}] = \boxed{\frac{1}{n}}$

or

$$= \int_0^\infty x \underbrace{n e^{-nx}}_{Exp(\frac{1}{n})\ pdf} dx = \frac{1}{n}\underbrace{\int_0^\infty v e^{-v} dv}_{E(W),\ W \sim Exp(1)} = \frac{1}{n}\int_0^\infty w e^{-w} dw$$

$(v = nx,\ dv = n\,dx)$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \frac{1}{n}\left[\int u\,dv - \int v\,du\right] = \left[-x e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})dx\right]\frac{1}{n} = \left[\int_0^\infty e^{-x}dx\right]\frac{1}{n}$$

$$= \frac{1}{n}\left[-e^{-x}\Big|_0^\infty\right] = \frac{1}{n}\left[0 - (-1)\right] = 1\,\frac{1}{n} = \frac{1}{n}$$

5) Let $X_1, \ldots, X_n$ be independent identically distributed random variables with pdf

$$f(x) = \sqrt{\frac{\sigma}{2\pi x^3}} \exp\left(-\frac{\sigma}{2x}\right)$$

$\sum \frac{1}{x_i} \ne \frac{1}{\sum x_i}$ $\qquad \prod_{i=1}^n e^{a_i} = e^{\sum_{i=1}^n a_i}$

where x and σ are both positive. Find a sufficient statistic $T(\boldsymbol{X})$ for σ.

$$f(x) = \underbrace{\sqrt{\frac{\sigma}{2\pi}}}_{c(\sigma)} \underbrace{\left(\frac{1}{\sqrt{x^3}}\right) I(x > 0)}_{h(x)} \exp\left(-\underbrace{\frac{\sigma}{2}}_{w(\sigma)} \underbrace{\frac{1}{x}}_{t(x)}\right)$$

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or

$$f(\underline{x}) = \underbrace{\prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi x_i^3}} I(x_i > 0)\right]}_{h(\underline{x})} \underbrace{\sigma^{n/2} \exp\left(-\frac{\sigma}{2}\sum_{i=1}^n \frac{1}{x_i}\right)}_{g(T(\underline{x})|\sigma)}$$

$$= \overbrace{\frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{\prod x_i^{3/2}} I(x_{(1)} > 0)} \ \overbrace{\sigma^{n/2} \exp\left(-\frac{\sigma}{2}\sum_{i=1}^n \frac{1}{x_i}\right)}$$

So $T(\underline{X}) = \sum_{i=1}^n \frac{1}{X_i}$ is suff

$\underbrace{\frac{1}{\lambda}\phi}_{c(\lambda)} \underbrace{x^{\phi-1} I(x>0)}_{h(x)} \exp[\underbrace{\lambda}_{w(\lambda)} \underbrace{\log(1+x^\phi)}_{t(y)}]$

I(x>0) or -1

6) Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(x) = \underbrace{\frac{1}{\lambda}}_{c(\lambda)} \underbrace{\phi x^{\phi-1} \frac{1}{1+x^\phi} I(x>0)}_{h(x)} \exp\left[\underbrace{-\frac{1}{\lambda}}_{w(\lambda)} \underbrace{\log(1+x^\phi)}_{t(y)}\right]$$

where x, ϕ, and λ are all positive. ASSUME ϕ IS KNOWN. find a complete sufficient statistic for λ.

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so $\boxed{\sum_{i=1}^{n} \log(1+X_i^\phi)}$ is complete

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7) Let $Y_1, ..., Y_n$ be iid with pdf $f(y) = \theta \exp[(\theta-1)\log(y)]I(0 < y < 1)$ where $\theta > 0$. Find a minimal sufficient statistic for θ using the the Lehmann Scheffé LSM theorem. **(Do not use REF theory.)**

$$R_{\underline{x},\underline{y}}(\theta) = \frac{f(\underline{x})}{f(\underline{y})} = \frac{\exp[(\theta-1)\sum\log(x_i)]\, I[0 < x_{(1)} < x_{(n)} < 1]}{\exp[(\theta-1)\sum\log(y_i)]\, \underbrace{I(0 < y_{(1)} < y_{(n)} < 1)}_{\text{constant wrt } \theta}} =$$

$= c \;\; \forall \theta > 0$

iff $\exp\left((\theta-1)\left[\sum\log(x_i) - \sum\log(y_i)\right]\right) = d \;\; \forall \theta > 0$

iff $\sum\log(x_i) = \sum\log(y_i)$

so $\boxed{\sum_{i=1}^{n} \log(Y_i) \text{ is min suff}}$

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