Math 580 2017 Exam 3 Name___________

Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf

$$f(x) = \frac{1}{\lambda} x^{\frac{1}{\lambda}-1}, \qquad = \left(\frac{1}{\lambda}\right) \exp\left[\left(\frac{1}{\lambda}-1\right)\log(x)\right]$$

where $\lambda > 0$ and $0 \leq x \leq 1$.

10 a) Find the maximum likelihood estimator of $\lambda$. (Make sure that you prove that your answer is the MLE.) $L(\lambda) = \frac{1}{\lambda^n} \exp\left[\left(\frac{1}{\lambda}-1\right)\sum x_i\right] = \frac{1}{\lambda^n} \prod_{i=1}^{n} x_i^{\frac{1}{\lambda}-1}$

$$\log L(\lambda) = -n \log \lambda + \left(\frac{1}{\lambda}-1\right) \sum \log x_i$$

$$\frac{d \log L(\lambda)}{d\lambda} = \frac{-n}{\lambda} - \frac{1}{\lambda^2} \sum \log x_i \overset{set}{=} 0$$

or $n\lambda = -\sum \log x_i$ or $\boxed{\hat{\lambda} = \frac{-\sum \log X_i}{n}}$, unique

$$\frac{d^2}{d\lambda^2} \log L(\lambda) = \frac{n}{\lambda^2} + \frac{2\sum \log X_i}{\lambda^3}\Bigg|_{\hat{\lambda}} = \frac{n}{\hat{\lambda}^2} - \frac{2n\hat{\lambda}}{\hat{\lambda}^3} = \frac{-n}{\hat{\lambda}^2} < 0$$

9 b) What is the maximum likelihood estimator of $\lambda^3$ ? Explain.

$$\boxed{\hat{\lambda}^3 = \left(\frac{-\sum \log X_i}{n}\right)^3}$$

by invariance

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$E(Y) = p \qquad V(Y) = 2p$

2) Let $Y_1, ..., Y_n$ be iid chi-square $\chi^2_p$ random variables.

a) Find the limiting distribution of $\sqrt{n}(\ \overline{Y}_n - c\ )$ for appropriate constant $c$

$\sqrt{n}(\overline{Y} - p) \xrightarrow{D} N(0, 2p)$ by CLT

b) Find the limiting distribution of $\sqrt{n}\ [(\overline{Y}_n)^{\frac{1}{2}} - d\ ]$ for appropriate constant $d$.

$g(p) = p^{\frac{1}{2}} \qquad g'(p) = \frac{1}{2} p^{-\frac{1}{2}} = \frac{1}{2p^{1/2}}, \ [g'(p)]^2 = \frac{1}{4p}$

$\sqrt{n}\left((\overline{Y}_n)^{\frac{1}{2}} - p^{\frac{1}{2}}\right) \xrightarrow{D} N\left(0, 2p\frac{1}{4p}\right) = \boxed{N\left(0, \frac{1}{2}\right)}$

by delta method

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3) Let $Y_1, ..., Y_n$ be iid exponential($\lambda$) random variables. Let $T = c\overline{Y}$ be an estimator of $\lambda$ where $c$ is a constant. a) Find the mean square error (MSE) of $T$ as a function of $c$ (and of $\lambda$ and $n$).

$E(T) = cE\overline{Y} = cEY = c\lambda$

$V(T) = c^2 V(\overline{Y}) = c^2 \frac{V(Y)}{n} = \frac{c^2\lambda^2}{n}$

$MSE(T) = V(T) + [B(T)]^2 =$

$\boxed{\frac{c^2\lambda^2}{n} + (c\lambda - \lambda)^2 = \frac{c^2\lambda^2}{n} + [\lambda(c-1)]^2}$

$= \lambda^2 \left[\frac{c^2}{n} + (c-1)^2\right]$

b) What value of $c$ makes $T$ an unbiased, consistent estimator of $\lambda$?

$\boxed{c = 1}$ so $T = \underbrace{\overline{Y}}_{\text{WLLN}}.$ Then $MSE = \frac{\lambda^2}{n} \to 0.$



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$f(y) = \frac{\theta^y e^{-\theta}}{y!}$ $\theta$ $\theta$

4) Let $Y_1, ..., Y_n$ be iid Poisson($\theta$) random variables where $\lambda > 0$.
a) Find the (Fisher) information number $I_1(\theta)$.

$$\log f(y) = \log \frac{1}{y!} + y \log \theta - \theta$$

$$\frac{\partial \log f(y)}{\partial \theta} = \frac{y}{\theta} - 1, \quad \frac{\partial^2 \log f(y)}{\partial \theta^2} = \frac{-y}{\theta^2}$$

$$I_1(\theta) \overset{\text{1PREF}}{=} -E_\theta\left[\frac{-Y}{\theta^2}\right] = \frac{\theta}{\theta^2} = \boxed{\frac{1}{\theta}}$$

b) Find the FCRLB for the variance of an unbiased estimator of $\tau(\theta) = \theta^2$. $\theta^2 = \tau(\theta)$

$$\tau'(\theta) = 2\theta$$

$$\text{FCRLB} = \frac{[\tau'(\theta)]^2}{n I_1(\theta)} = \frac{4\theta^2}{n \frac{1}{\theta}} = \boxed{\frac{4\theta^3}{n}}$$

c) Find the UMVUE of $\theta^2$. Hint: the UMVUE $= a(T)^2 + bT$ where $T$ is the UMVUE of $\lambda$.

$$\theta^2 = E\left[a(\bar{Y})^2 + b\bar{Y}\right] = aE(\bar{Y})^2 + b\theta$$

$$= a\left[V(\bar{Y}) + (E(\bar{Y}))^2\right] + b\theta$$

$$= a\left(\frac{\theta}{n} + \theta^2\right) + b\theta$$

$$= \frac{a\theta}{n} + a\theta^2 + b\theta.$$

Take $a = 1$ and $b = \frac{-1}{n}$.

Then $E\left[(\bar{Y})^2 - \frac{1}{n}\bar{Y}\right] = \theta^2$

so $\boxed{(\bar{Y})^2 - \frac{1}{n}\bar{Y}}$ is the UMVUE of $\theta^2$ by LSU.

$$= \frac{(\sum Y_i)^2 - \sum Y_i}{n^2}$$

T UMVUE of $\theta$ and invariance $-5$

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5) Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf

$$f(y) = \sqrt{\frac{\sigma}{2\pi y^3}} \exp\left(\frac{-\sigma}{2y}\right)$$

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where $y$ and $\sigma$ are both positive. You may use the fact that $W = 1/Y \sim G(1/2, 2/\sigma)$.

a) Find a complete sufficient statistic $T(\boldsymbol{Y})$ for $\sigma$.

$$f(y) = \underbrace{\sqrt{\frac{1}{2\pi y^3}} I(y > 0)}_{h(y)} \underbrace{\sqrt{\sigma}}_{c(\sigma)} \exp\left(\underbrace{\frac{\sigma}{2}}_{w(\sigma)\uparrow} \underbrace{\frac{-1}{y}}_{t(y)}\right) \qquad \Omega = (0, \infty)$$

1PREF so $T(\underline{Y}) = \sum_{i=1}^n \frac{1}{Y_i}$ or $T(\underline{Y}) = -\sum_{i=1}^n \frac{1}{Y_i}$

b) Find the UMP level $\alpha$ test for $H_0 : \sigma = 1$ versus $H_A : \sigma > 1$. Hint: make sure $w(\sigma)$ is an increasing function.

So $T(\underline{Y}) = -\sum_{i=1}^n \frac{1}{Y_i}$

reject $H_0$ if $T(\underline{Y}) > k$

where $P_1(T(\underline{Y}) > k) = \alpha$ — stop

$P_1\left(-\sum_{i=1}^n \frac{1}{Y_i} > k\right) = \alpha$

so $P_1\left(\sum_{i=1}^n \frac{1}{Y_i} < -k\right) = \alpha$

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