$\theta\int_0^1 (1-y)^{\theta-1}dy = -\theta\int_1^0 v^{\theta-1}dv = \theta\int_0^1 v^{\theta-1}dv = \theta\frac{v^\theta}{\theta}\Big|_0^1 = 1$

$v = 1-y \quad dv = -dy$

room 208?

Math 580 2017 Final Name___________

1) Let $Y_1, ..., Y_n$ be a random sample from a distribution with pdf

beta ($\delta = 1, \nu = \theta$)

$$f(y) = \theta(1-y)^{\theta-1} I(0 < y < 1)$$

where $\theta > 0$.

a) Find the maximum likelihood estimator of θ. (Make sure that you prove that your answer is the MLE.)

$L(\theta) = \prod_{i=1}^{n} f(y_i) = \theta^n \prod (1-y_i)^{\theta-1} = \theta^n \exp\left[(\theta-1)\sum \log(1-y_i)\right]$

$\log L(\theta) = n \log\theta + (\theta-1)\sum \log(1-y_i)$

$\frac{d \log L(\theta)}{d\theta} = \frac{n}{\theta} + \sum \log(1-y_i) \overset{set}{=} 0$

or $n = -\theta \sum \log(1-y_i)$

$$\boxed{\hat{\theta} = \frac{-n}{\sum \log(1-y_i)}}$$, unique

or -4

$\frac{d^2 \log L(\theta)}{d\theta^2} = \frac{-n}{\theta^2} < 0$

b) What is the maximum likelihood estimator of $1/\theta^2$? Explain.

$\frac{1}{\hat{\theta}^2} = \left(\frac{\sum \log(1-y_i)}{-n}\right)^2 = \left(\frac{\sum \log(1-y_i)}{n}\right)^2$

by invariance

40

JQUAL 8:18

beta(r=1, θ)

2) Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf (probability density function)

1PREF or-1

$$f(y|\theta) = \theta y^{\theta-1} I(0 < y < 1), \quad \theta > 0. \quad = \underbrace{\theta}_{c(\theta)} \underbrace{I(0<y<1)}_{h(y)} \exp[(\theta-1)\log y]$$

Then $-\log(Y) \sim$ exponential$(1/\theta)$.

a) Find $I_1(\theta)$. $\log f(y|\theta) = \log\theta + (\theta-1)\log y$

$$\frac{d}{d\theta}\log f(y|\theta) = \frac{1}{\theta} + \log y$$

$$\frac{d^2}{d\theta^2}\log f(y|\theta) = \frac{-1}{\theta^2}$$

1PREF

$$I_1(\theta) \overset{\downarrow}{=} -E\left(\frac{-1}{\theta^2}\right) = \boxed{\frac{1}{\theta^2}}$$

b) Find the FCRLB for unbiased estimators of θ^2. $\tau(\theta) = \theta^2, \ \tau'(\theta) = 2\theta$

$$= \frac{[\tau'(\theta)]^2}{n I_1(\theta)} = \frac{4\theta^2}{n\frac{1}{\theta^2}} \qquad \boxed{\frac{4\theta^4}{n}}$$

4

9

got it

c) Find the uniformly minimum unbiased estimator (UMVUE) of $1/\theta^2$. (Hint: find $E(T_n^2)$ where T_n is the UMVUE of $1/\theta$.) Let $W_i = -\log Y_i$. $EW = \frac{1}{\theta}$, $V(W) = \frac{1}{\theta^2}$

$$T_n = \overline{W} = \frac{-1}{n}\sum \log(Y_i) \sim \frac{1}{n}G(n, \tfrac{1}{\theta}). \quad \left(E\overline{W} = \frac{1}{n} n \frac{1}{\theta} = \frac{1}{\theta},\ V(\overline{W}) = \frac{1}{n^2}\frac{n}{\theta^2} = \frac{1}{n\theta^2}\right)$$

$$ET_n^2 = E(\overline{W})^2 = V(\overline{W}) + (E\overline{W})^2 = \frac{V(W)}{n} + (EW)^2$$

$$= \frac{1}{n\theta^2} + \frac{1}{\theta^2}\frac{n}{n} = \frac{n+1}{n\theta^2} \quad \text{so}$$

$$\boxed{\frac{n}{n+1}T_n^2 = \frac{n}{n+1}(\overline{W})^2 = \frac{n}{n+1}\left(\frac{-\sum\log Y_i}{n}\right)^2 = \frac{1}{n(n+1)}\left(-\sum\log Y_i\right)^2}$$ is the UMVUE by LSU.

60

$w(\lambda)\uparrow$ } or -6   IW($\phi=1, \lambda$)

3) Let $Y_1, ..., Y_n$ be iid with pdf

$$f(y) = \frac{1}{\lambda} \frac{1}{y^2} \exp\left(\frac{-1}{\lambda} \frac{1}{y}\right)$$

1PREF

where $y > 0$ and $\lambda > 0$. You may use the fact that $1/Y \sim EXP(\lambda) \sim \frac{\lambda}{2}\chi^2_2$.

e   a) Find the UMP level α test for $H_0 : \lambda = 1$ versus $H_1 : \lambda = 1.318$.

reject H_0 if $\sum_{i=1}^{n} \frac{1}{y_i} > k$ where $\alpha = P_1\left(\sum_{i=1}^{n} \frac{1}{Y_i} > k\right)$

(hard way: or NP $\frac{f(\underline{y}|\theta_1)}{f(\underline{y}|\theta_0)} = \ldots > k'$ iff $\sum \frac{1}{y_i} > k$)

$\frac{5}{9}$ got it   b) Suppose $n = 10$ and $\alpha = 0.05$. Find the power $\beta(1.318)$ when $\lambda = 1.318$ using the chi-square table.

χ^2 table gives area to right

	.05
20	31.41

$$\alpha = 0.05 = P_1\left(\sum_{i=1}^{10} \frac{1}{Y_i} > k\right) = P\left(2\sum_{i=1}^{10} \frac{1}{Y_i} > 2k\right)$$

$$\left(\sum_{i=1}^{n} t(Y_i) \sim \frac{\lambda}{2}\chi^2_{2n} \text{ so } \frac{2}{\lambda}\sum_{i=1}^{n} t(Y_i) \sim \chi^2_{2n}\right)$$

$$= P(\chi^2_{20} > 2k) \text{ so } 2k = \chi^2_{20,.95} = 31.41$$

$$\text{or } k = \frac{31.41}{2} = 15.705$$

$$\text{so } \beta(1.318) = P_{1.318}\left(\sum \frac{1}{Y_i} > 15.705\right) = P\left(\frac{2}{1.318}\sum \frac{1}{Y_i} > \frac{2(15.705)}{1.318}\right)$$

$$= P(\chi^2_{20} > 23.832) = \boxed{0.25}$$

df	.25
20	23.83

#0

Qual Sept 19

hard way $V(Y) = \sigma^2\left(3 - \frac{8}{\pi}\right)$ with difficulty so $\frac{S_n^2}{\left(3-\frac{8}{\pi}\right)^2}$ works

MB $(\mu = 0, \sigma^2)$

4) Suppose $Y_1, ..., Y_n$ are iid with pdf

$$\sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma^3}\ y^2 I(y>0)\ \exp\left(\frac{-1}{2\sigma^2}\ y^2\right)$$

where $\sigma^2 > 0$ and $W = Y^2 \sim G(3/2, 2\sigma^2)$.

a) The method of moments estimator of $\mu_2 = E(Y^2)$ is $\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^n Y_i^2$. Find $\mu_2 = E(Y^2)$.

$$= E(W) = \frac{3}{2}\, 2\sigma^2 = \boxed{3\sigma^2}$$

b) Find the method of moments estimator of σ^4.

$\sigma^4 = g(\mu_2) = g(3\sigma^2) = \frac{(3\sigma^2)^2}{9} = \frac{\mu_2^2}{9} = \left(\frac{\mu_2}{3}\right)^2$

This is also MLE.

$$\left(\frac{\hat{\mu}_2}{3}\right)^2 = \left(\frac{1}{3n}\sum_{i=1}^n Y_i^2\right)^2 = \frac{\hat{\mu}_2^2}{9} = \frac{1}{9}\left(\frac{\sum Y_i^2}{n}\right)^2$$

c) Find the limiting distribution of $\sqrt{n}(\ \overline{W}_n - c\ )$ for appropriate constant c.

$EW = 3\sigma^2 \qquad V(W) = \frac{3}{2}\, 4\sigma^4 = 6\sigma^4$

$$\sqrt{n}\left(\overline{W}_n - 3\sigma^2\right) \xrightarrow{D} N(0, 6\sigma^4)$$ by the CLT

$\frac{1}{9}$ got it → d) Find the limiting distribution of $\sqrt{n}\ [(\overline{W}_n)^2 - d\ ]$ for appropriate constant d.

$\theta = 3\sigma^2, \quad g(\theta) = \theta^2, \quad g'(\theta) = 2\theta = 6\sigma^2, \quad [g'(\theta)]^2 = 4\theta^2 = 4(3\sigma^2)^2 = 36\sigma^4$

$$\sqrt{n}\left((\overline{W}_n)^2 - 9\sigma^4\right) \xrightarrow{D} N(0, 36\sigma^4\ 6\sigma^4)$$

$$= N(0, 216\sigma^8)$$

by the delta method

80

$f(y) = \frac{1}{\lambda} e^{-\frac{1}{\lambda} y}$ $I(y>0)$ MLE $\bar{y}$

$L(\lambda) = \frac{1}{\lambda^n} e^{-\frac{1}{\lambda}\sum y_i}$

5) Let $Y_1, ..., Y_n$ be iid exponential (λ) random variables where $\lambda > 0$.

$\frac{1}{4}$ got it → a) Find the α level likelihood ratio test for $H_0 : \lambda = 1$ vs $H_1 : \lambda \neq 1$. (If you know the MLE of λ, then you do not need to derive it.)

$$\lambda(\underline{y}) = \frac{L(\hat{\lambda}_0)}{L(\hat{\lambda})} = \frac{e^{-\sum y_i}}{\frac{1}{(\bar{y})^n} e^{-\frac{1}{\bar{y}}\sum y_i}} = \frac{(\bar{y})^n e^{-n\bar{y}}}{e^{-n}} = \frac{(\bar{y})^n e^{-\sum y_i}}{e^{-n}}$$

reject Ho if $\lambda(\underline{y}) \leq c$ where $\alpha = P_1(\lambda(\underline{Y}) \leq c)$

b) If $\lambda(\boldsymbol{y})$ is the LRT test statistic of the above test, use the approximation

$\frac{1}{4}$ got it

$$-2 \log \lambda(\boldsymbol{y}) \approx \chi^2_d$$

for the appropriate degrees of freedom d to find the rejection region of the test **in useful form** if $\alpha = 0.05$.

$d = 1$ or at least $+10$

reject Ho if $-2\log \lambda(\underline{y}) > \chi^2_{1, 0.95} = 3.84$

where $P(-2\log \lambda(\underline{Y}) > 3.84) \approx .05 = \alpha$

df	.05
1	3.84

AUS Q17d6

40

$-2\log \lambda > 3.84$ iff $\log \lambda < 1.92$ iff $\lambda \leq 6.82$

see Sept19 qual

MB($\mu = 0, \sigma^2$)

6) Suppose $Y_1, ..., Y_n$ are iid with pdf

$$\sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma^3}\ y^2 I(y > 0)\ \ \exp\left(\frac{-1}{2\sigma^2}\ y^2\right)$$

where $\sigma^2 > 0$ and $W = Y^2 \sim G(3/2, 2\sigma^2)$.

e a) Find a complete sufficient statistic for σ^2. 1PREF

So $\boxed{\sum_{i=1}^n Y_i^2}$

b) Let $T_n = c\overline{W}_n$ be an estimator of σ^2. Find the mean square error (MSE) of T_n.

$$E T_n = c E \overline{W}_n = c E W = c \frac{3}{2} 2\sigma^2 = 3c\sigma^2$$

$$V(T_n) = c^2 V(\overline{W}) = c^2 \frac{V(W)}{n} = c^2 \frac{\frac{3}{2}(4\sigma^4)}{n} = \frac{6c^2\sigma^4}{n}$$

$$MSE(T_n) = V(T_n) + (E T_n - \sigma^2)^2$$

$$\boxed{= \frac{6c^2\sigma^4}{n} + (3c\sigma^2 - \sigma^2)^2 = \frac{6c^2\sigma^4}{n} + \sigma^4 (3c-1)^2}$$

40