Math 580 Quiz 10 Spring 2017 Name____________________

1) Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf

$$f(x) = \frac{1}{\lambda}\phi x^{\phi-1}\frac{1}{1+x^{\phi}}\exp\left[-\frac{1}{\lambda}\log(1+x^{\phi})\right]$$

where $x, \phi$, and $\lambda$ are all positive. ASSUME $\phi$ IS KNOWN. If possible, find the UMP (uniformly most powerful) level $\alpha$ test of $Ho: \lambda = 1$ versus $H_1: \lambda < 1$. (Note that the inequality in $H_1$ is "less than.") 1PREF with $w(\lambda)\uparrow$ so UMP test is

reject Ho if $\sum \log(1+x_i^{\phi}) < k$ where $\alpha = P_1(\sum \log(1+x_i^{\phi}) < k)$,

or NP $\frac{f(\underline{x},\lambda_1)}{f(\underline{x},1)} = \frac{\frac{1}{\lambda_1^n}\exp\left[\frac{-1}{\lambda_1}\sum\log(1+x_i^{\phi})\right]}{\exp\left[-\sum\log(1+x_i^{\phi})\right]} > k$ iff

$\exp\left[\underbrace{\left(\frac{-1}{\lambda_1}+1\right)}_{<0 \text{ if } \lambda_1<1}\sum\log(1+x_i^{\phi})\right] > k'$ iff $\sum\log(1+x_i^{\phi}) < c$ so same as boxed test

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2) If $\sum_{i=1}^n Y_i$ is a complete sufficient statistic for $\theta$, then by the LSU theorem, $e^{t\sum_{i=1}^n Y_i}$ is the UMVUE of $E[e^{t\sum_{i=1}^n Y_i}] = m_{\sum_{i=1}^n Y_i}(t)$, the mgf of $\sum_{i=1}^n Y_i$. If $Y_1, ..., Y_n$ are iid Poisson$(\theta)$, then $e^{t\sum_{i=1}^n Y_i} = \exp(t\sum_{i=1}^n Y_i)$ is the UMVUE of $\tau(\theta)$. Find $\tau(\theta)$.

$\sum Y_i \sim Pois(n\theta)$ has mgf $E(\exp t\sum Y_i)$

$= [m_{Y_1}(t)]^n = [e^{\theta(e^t-1)}]^n = \boxed{e^{n\theta(e^t-1)} = \tau(\theta) = \exp[n\theta(e^t-1)]}$

$\left(t = \log(1+\frac{1}{n}) = \log(\frac{n+1}{n}) \Rightarrow \tau(\theta) = e^{\theta}\right)$

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$\frac{\theta}{2} \exp\left[-(\theta+1)\log(1+|y|)\right]$ is a 1PREF

3) Let $Y_1, ..., Y_n$ be iid random variables from a distribution with pdf

$$f(y) = \frac{\theta}{2(1+|y|)^{\theta+1}}$$

Key trick

$\text{EXP}(\frac{1}{\theta}) = G(1, \frac{1}{\theta})$

where $\theta > 0$ and $y$ is real. Then $W = \log(1+|Y|)$ has pdf $f(w) = \theta e^{-w\theta}$ for $w > 0$.

a) Find the (Fisher) information number $I_1(\theta)$.

$\log \frac{\theta}{2} = \log\theta - \log 2$

$$\log f(y|\theta) = \log(\tfrac{\theta}{2}) - (\theta+1)\log(1+|y|)$$

$$\frac{d}{d\theta}\log f(y|\theta) = \frac{1}{\theta} - \log(1+|y|)$$

$\left(\frac{d}{d\theta}\log\frac{\theta}{2} = \frac{1}{\frac{\theta}{2}}\,\frac{1}{2} = \frac{1}{\theta}\right)$

$$\frac{d^2}{d\theta^2}\log f(y|\theta) = \frac{-1}{\theta^2}$$

1PREF

$$I_1(\theta) = -E\left(\frac{-1}{\theta^2}\right) = \boxed{\frac{1}{\theta^2}}$$

b) Find the uniformly minimum variance unbiased estimator (UMVUE) for $1/\theta$.

$T = \sum_{i=1}^{n} \log(1+|Y_i|) \sim G(n, \frac{1}{\theta})$ since $W \sim \text{EXP}(\frac{1}{\theta})$

so $E(T) = n\frac{1}{\theta} = n E(W)$

So $\boxed{\frac{T}{n} = \frac{\sum \log(1+|Y_i|)}{n} \text{ is the UMVUE of } \frac{1}{\theta}}$ by LSU

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4) Let $Y_1, ..., Y_n$ be iid exponential ($\lambda$) random variables. Find the UMP level $\alpha$ test for $H_0: \lambda = 1$ versus $H_1: \lambda = 10$.

$f(y) = I(y>0)\,\frac{1}{\lambda}\exp\left(-\frac{1}{\lambda}y\right)$ 1PREF

$w(\lambda)\uparrow$

So $\boxed{\text{reject } H_0 \text{ if } \sum_{i=1}^{n} Y_i > k \text{ where } \alpha = P_1\left(\sum_{i=1}^{n} Y_i > k\right)}$

or 3

(or $\frac{f(y|10)}{f(y|1)} = \frac{(\frac{1}{10})^n \exp(-\frac{1}{10}\sum y_i)}{\exp(-\sum y_i)} = (\frac{1}{10})^n \exp\left[(1-\frac{1}{10})\sum y_i\right] > k'$

$>1$

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iff $\sum Y_i > k$ so boxed answer.