22] If you take the qual, memorize the solution to 5.35 MLE of $N(a\mu, \mu)$ data.

Also know how to find the MLE of $N(\mu, \mu^2)$ data.

skip ex 5.4

3rd big qual problem for final

§5.2 23] know p141 - Let $\hat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n} Y_i^j$ and

$\mu_j = \mu_j(\underset{\sim}{\theta}) = \mu_j(\theta_1, \ldots, \theta_k) = E_{\underset{\sim}{\theta}} Y^j$. Solve the system

$$\hat{\mu}_1 \overset{set}{=} \mu_1(\theta_1, \ldots, \theta_k)$$
$$\vdots$$
$$\hat{\mu}_k \overset{set}{=} \mu_k(\theta_1, \ldots, \theta_k) \quad \text{for} \quad \hat{\underset{\sim}{\theta}}_{MM},$$

the method of moments estimator of $\underset{\sim}{\theta}$.

If g is a continuous function of the 1st k moments, and $h(\underset{\sim}{\theta}) = g(\mu_1(\underset{\sim}{\theta}), \ldots, \mu_k(\underset{\sim}{\theta}))$, then

the method of moments estimator of $h(\underset{\sim}{\theta})$ is $g(\hat{\mu}_1, \ldots, \hat{\mu}_k)$.

ex] Typically $k=1$ or $k=2$.

i) If $E_\theta(Y) = h(\theta) = \mu_1(\theta)$ then $\bar{Y} \overset{set}{=} h(\theta)$ has solution $\hat{\theta}_{MM} = h^{-1}(\bar{Y})$.

ii) $V_\theta(Y) = \mu_2(\theta) - [\mu_1(\theta)]^2$

The method of moments estimator of $V_\theta(Y)$ is $S_m^2 = \frac{1}{n}\sum^{n} Y_i^2 - (\bar{Y})^2 = \frac{1}{n}\sum_{i=1}^{n} (Y_i - \bar{Y})^2$.

Note that S^2_M is also the MLE of σ^2 if $Y_1, \ldots, Y_n$ are iid $N(\mu, \sigma^2)$ with μ and σ^2 unknown

24) p 142 <u>plug in principle</u>: if $E_{\underline{\theta}}(Y) = h_1(\theta_1, \theta_2)$ and $V_{\underline{\theta}}(Y) = h_2(\theta_1, \theta_2)$,

solving $\overline{Y} \stackrel{set}{=} h_1(\theta_1, \theta_2)$
$S_m^2 \stackrel{set}{=} h_2(\theta_1, \theta_2)$ for $\hat{\underline{\theta}}_{MM}$ yields a method of moments estimator.

ex] Read ex 5.10.

ch 6 1] know p 157 Let $\underline{Y} = (Y_1, \ldots, Y_n)$ have pdf or pmf $f(\underline{y}|\underline{\theta})$ for $\underline{\theta} \in \Theta$. Let $\tau(\underline{\theta})$ be a real valued function of $\underline{\theta}$ and let $T(\underline{Y})$ be an estimator of $\tau(\underline{\theta})$. The <u>bias</u> of an estimator $T(\underline{Y})$ for $\tau(\underline{\theta})$ is

test statistic

$$B(T) \equiv B_{\tau(\underline{\theta})}(T) = E_{\underline{\theta}}(T) - \tau(\underline{\theta}).$$

The mean square error <u>MSE</u> of an estimator $T(\underline{Y})$ for $\tau(\underline{\theta})$ is

$$MSE(T) \equiv MSE_{\tau(\underline{\theta})}(T) = E_{\underline{\theta}}[(T - \tau(\underline{\theta}))^2]$$

$$= V_{\underline{\theta}}(T) + [B_{\tau(\underline{\theta})}(T)]^2$$

2] p 157 T is an unbiased estimator of $\tau(\underline{\theta})$ if $E_{\underline{\theta}}(T) = \tau(\underline{\theta})$ $\forall\ \underline{\theta} \in \Theta$.

3] Know The 4th main type of question on the 3rd midterm, final and qual is defining a class of estimators $\{T_k(\underline{Y}), k \in \Lambda\}$ and finding k_0 such that $T_{k_0}(\underline{Y})$

minimizes the MSE. Typically write MSE as a function of k and find the 1st and 2nd derivatives like the MLE problem, but you are looking for a global minimizer.

See ex 6.1, 6.2, 6.3

ex] Jan 2000 Qual $X_1, \ldots, X_n$ iid
$f(x|\theta) = e^{-(x-\theta)}, \ x > \theta.$ $T_a = X_{(1)} + a.$
Find a that minimizes the MSE if T_a is an estimator of θ.

Soln $P(X_{(1)} \le t) = 1 - P(X_{(1)} > t) = 1 - [P(X_1 > t)]^n$
$= 1 - [1 - F(t)]^n.$ Now
$F(t) = \int_\theta^t e^{-(x-\theta)} dx = e^\theta \int_\theta^t e^{-x} dx$
$= e^\theta (-e^{-x}) \Big|_\theta^t = e^\theta (-e^{-t} + e^{-\theta}) = 1 - e^{\theta} e^{-t}, \ t > \theta$

better! use formula for this

So $P(X_{(1)} \le t) = 1 - [1 - (1 - e^\theta e^{-t})]^n = 1 - [e^{-(t-\theta)}]^n$
$= 1 - e^{-n(t-\theta)}, \ t > \theta.$

So $f_{X_{(1)}}(t) = \underbrace{n e^{-n(t-\theta)}}_{\text{location family with standard pdf } n e^{-nt}, \ t > 0}, \ t > \theta.$

So $E(X_{(1)}) = E(\underset{\underset{EXP(\frac{1}{n})}{\uparrow}}{\theta + Z}) = \theta + \frac{1}{n}$,

or use $E[X_{(1)}] = \int_\theta^\infty t\, n\, e^{-n(t-\theta)}\, dt$

$u = t-\theta$, $du = dt$, $t = u+\theta$, $t=\theta \to u=0$, $t=\infty \to u=\infty$

$$= \int_0^\infty (u+\theta)\, n e^{-nu} du = \theta \underbrace{\int_0^\infty n e^{-nu} du}_{1 = \int EXP(\frac{1}{n}) \text{ pdf}} + \underbrace{\int_0^\infty u\, n e^{-nu} du}_{\frac{1}{n} = E[EXP(\frac{1}{n}) \text{ RV}]}$$

(or use integration by parts)

$= \theta + \frac{1}{n}$.

So $MSE(T_a) = E_\theta (X_{(1)} + a - \theta)^2 =$

$V_\theta (X_{(1)} + a) + (E_\theta[X_{(1)} + a] - \theta)^2$

$$= \overbrace{V_\theta(X_{(1)})}^{\text{free of } a} + \overbrace{(E_\theta(X_{(1)}) + a - \theta)^2}^{\text{minimize this}}$$

$= V_\theta(X_{(1)}) + (\theta + \frac{1}{n} + a - \theta)^2$

$= V_\theta(X_{(1)}) + (\frac{1}{n} + a)^2$.

Take $a = -\frac{1}{n}$. Then $T_a = X_{(1)} - \frac{1}{n}$ minimizes the MSE.

§6.2 4) Know p160 Let $\underline{Y}$ have pmf or pdf $f(\underline{y}|\theta)$ for $\theta \in \Theta$. Then $U \equiv U(\underline{Y})$ is the uniformly minimum variance unbiased estimator UMVUE of $\tau(\theta)$ if U is an unbiased estimator of $\tau(\theta)$ and if $V_\theta(U) \le V_\theta(W)$ $\forall\, \theta \in \Theta$ where W is any other unbiased estimator of $\tau(\theta)$.

5] Know p160 Lehmann Scheffé LSU theorem

If $T(\underline{Y})$ is a complete sufficient statistic, then $U = g(T(\underline{Y}))$ is the UMVUE of its expectation $E_\theta(U) = E_\theta[g(T(\underline{Y}))] = \tau(\theta)$.

In particular, if W is any unbiased estimator of $\tau(\theta)$, then $U \equiv g(T(\underline{Y})) = E[W(\underline{Y}) | T(\underline{Y})]$ (*) is the UMVUE of $\tau(\theta)$.

6] (*) is called Rao Blackwellization because of the next theorem.

7]* p160 <u>Rao Blackwell</u> (Lehmann Scheffé) <u>theorem</u>: Let $W \equiv W(\underline{Y})$ be an unbiased estimator of $\tau(\theta)$ and let $T \equiv T(\underline{Y})$ be a sufficient statistic for θ. Then $\phi(T) = E(W|T)$ does not depend on θ, $\phi(T)$ is an unbiased estimator of $\tau(\theta)$, and $V_\theta[\phi(T)] \le V_\theta(W) \; \forall \theta$.

proof. Assume variances exist. Since T is sufficient, $\underline{Y}|T$ and $W(\underline{Y})|T$ are free of θ, so $E_\theta(W|T) \equiv E(W|T) = \phi(T) \; \forall \theta \in \Theta$ and $\phi(T)$ is a statistic. Now $E_\theta[\phi(T)] = E_\theta[E(W|T)] = E_\theta(W) = \tau(\theta) \; \forall \theta \in \Theta$ by iterated expectations. So $\phi(T)$ is unbiased. By Steiner's formula,

$$V_\theta(W) = \underbrace{V_\theta(E(W|T))}_{V_\theta(\phi(T))} + \underbrace{E_\theta(V(W|T))}_{\ge 0} \ge V_\theta(\phi(T)) \; \forall \theta.$$

8]* One parameter exp families have a complete suff stat $T = \sum^n t(Y_i)$ if Ω contains an open interval

9] p162* If $\underline{Y}$ has pdf or pmf $f(\underline{y}|\theta)$, then the information number or Fisher Information is $I_{\underline{Y}}(\theta) \equiv I_n(\theta) = E_\theta\left(\left[\frac{\partial}{\partial\theta} \log f(\underline{Y}|\theta)\right]^2\right)$.

10] * p162 If $\eta = \tau(\theta)$ where $\tau'(\theta) \neq 0$, then $I_n(\eta) = I_n(\tau(\theta)) = \dfrac{I_n(\theta)}{[\tau'(\theta)]^2}$.

true for 1PREF

11] Know If $\frac{d}{d\theta}\int f(\underline{y}|\theta)d\underline{y} = \int \frac{d}{d\theta} f(\underline{y}|\theta)d\underline{y}$ and $Y_1, \ldots, Y_n$ are iid, $I_n(\theta) = n\, I_1(\theta)$.

12] Know p162-3 a) If $Y_1 \equiv Y$ is from a 1P-REF,

$$I_1(\theta) = -E_\theta\left[\frac{d^2}{d\theta^2} \log f(Y|\theta)\right].$$

b) If $Y_1, \ldots, Y_n$ are iid from a 1P-REF,
$I_n(\theta) = n\, I_1(\theta)$ and

$$I_n(\tau(\theta)) = \frac{n\, I_1(\theta)}{[\tau'(\theta)]^2}.$$

13] p164 know Let $\underline{Y}$ be data and consider $\tau(\theta)$ where $\tau'(\theta) \neq 0$. The Fréchet Cramér Rao lower bound (FCRLB or CRLB) is $FCRLB_n(\tau(\theta)) = \dfrac{[\tau'(\theta)]^2}{I_n(\theta)}$.

14] p164 know Th. Frechet Cramer Rao lower bound or information inequality: Let $Y_1, \ldots, Y_n$ be iid from a 1P-REF with pmf or pdf $f(y|\theta)$.

Let $w(\underline{Y})$ be any unbiased estimator of $\tau(\theta) \equiv E_\theta[w(\underline{Y})]$.

Then $V_\theta(w) \geq FCRLB_n(\tau(\theta)) = \dfrac{[\tau'(\theta)]^2}{I_n(\theta)} = \dfrac{[\tau'(\theta)]^2}{n\,I_1(\theta)}$.

If $\tau(\theta) = \theta$, $FCRLB_n(\theta) = \dfrac{1}{I_n(\theta)} = \dfrac{1}{n\,I_1(\theta)}$. Read Proof carefully.

ex} $Y \sim \text{gamma}(\nu, \lambda)$, ν known is a 1P REF.

$$f(y) = \frac{1}{\Gamma(\nu)} \frac{1}{\lambda^\nu} y^{\nu-1} e^{-y/\lambda}.$$

take log on support

$$\log f(y) = \log \frac{y^{\nu-1}}{\Gamma(\nu)} - \nu \log(\lambda) - \frac{y}{\lambda}$$

$$\frac{d \log f(y|\lambda)}{d\lambda} = -\frac{\nu}{\lambda} + \frac{y}{\lambda^2}$$

$$\frac{d^2 \log f(y|\lambda)}{d\lambda^2} = \frac{\nu}{\lambda^2} - \frac{2y}{\lambda^3}$$

same as for MLE

So $I_1(\lambda) \stackrel{\text{1PREF}}{\stackrel{\perp}{=}} -E_\lambda \dfrac{d^2 \log f(Y|\lambda)}{d\lambda^2} = -E_\lambda\left[\dfrac{\nu}{\lambda^2} - \dfrac{2Y}{\lambda^3}\right]$

$$= -\left[\frac{\nu}{\lambda^2} - 2\frac{\nu\lambda}{\lambda^3}\right] = \frac{\nu}{\lambda^2}.$$

hard way

or $I_1(\lambda) = E_\lambda\left[\left(\dfrac{d}{d\lambda} \log f(Y|\lambda)\right)^2\right] =$

$$E_\lambda\left[\frac{\nu^2}{\lambda^2} - \frac{2\nu Y}{\lambda^3} + \frac{Y^2}{\lambda^4}\right] = \frac{\nu^2}{\lambda^2} - \frac{2\nu\,\nu\lambda}{\lambda^3} + \frac{\nu\lambda^2 + \nu^2\lambda^2}{\lambda^4}$$

$$= \frac{\nu^2 - 2\nu^2 + \nu + \nu^2}{\lambda^2} = \frac{\nu}{\lambda^2}.$$

So $I_n(\lambda) = n\,I_1(\lambda) = n\nu/\lambda^2$.

$FCRLB_n(\lambda) = \dfrac{1}{n I_1(\lambda)} = \dfrac{\lambda^2}{n\nu}$, $FCRLB_n(\lambda^2) = \dfrac{[\tau'(\lambda)]^2}{n I_1(\lambda)} = \dfrac{(2\lambda)^2 \lambda^2}{n\nu} = \dfrac{4\lambda^4}{n\nu}$

ex] Let $X_1, \ldots, X_n$ be iid $N(\mu, \sigma^2)$, μ known, $\sigma^2 > 0$.

$$\underset{\text{1P REF}}{f(x)} = \underbrace{\frac{1}{\sqrt{2\pi}}}_{h(x)} \underbrace{\frac{1}{\sigma}}_{c(\sigma^2)} \exp\left[\underbrace{\frac{-1}{2\sigma^2}}_{w(\sigma^2)} \underbrace{(x-\mu)^2}_{t(x)}\right]. \quad \text{so}$$

$W = \sum_{i=1}^{n} (X_i - \mu)^2 \sim \sigma^2 \chi^2_n$ is complete suff.

From ch10, $E Y^k = \dfrac{2^k \, \Gamma(k + \frac{n}{2})}{\Gamma(\frac{n}{2})}$ if $Y \sim \chi^2_n$.

Thus $T_k(\underline{X}) = \dfrac{W^k}{E(Y^k)} = \dfrac{\Gamma(\frac{n}{2}) \, W^k}{2^k \, \Gamma(k + \frac{n}{2})}$ is the UMVUE of σ^{2k} for $k > 0$

(Since $E_{\sigma^2}(T_k) = \sigma^{2k} \dfrac{E Y^k}{E Y^k} = \sigma^{2k}$.)

Let $\theta = \sigma^2$ and $\tau_k(\theta) = \theta^k$ so $\tau_k'(\theta) = k \, \theta^{k-1}$.

The FCRLB for estimating $\tau_k(\theta) = \sigma^{2k}$ is

$$\frac{[\tau_k'(\sigma^2)]^2}{n \underbrace{I_1(\sigma^2)}_{\text{HW9 \#6}}} = \frac{[k \, \sigma^{2(k-1)}]^2}{n \frac{1}{2\sigma^4}} = \frac{2k^2 \sigma^{4k}}{n}.$$

Now $V_\theta[T_k(\underline{X})] = \left(\dfrac{1}{E Y^k}\right)^2 V_\theta(W^k) =$

$$\left(\frac{1}{E Y^k}\right)^2 \left[E_\theta W^{2k} - (E_\theta W^k)^2\right] \underset{\uparrow}{=} \sigma^{4k} \left[\frac{\Gamma(\frac{n}{2}) \Gamma(2k + \frac{n}{2})}{\Gamma(k + \frac{n}{2}) \Gamma(k + \frac{n}{2})} - 1\right]$$

details on HW

do this part

$= c_k \, \sigma^{4k}$. If $k = 1$, $c_1 =$

$$\frac{\Gamma(\frac{n}{2}) \Gamma(2 + \frac{n}{2})}{\Gamma(1 + \frac{n}{2}) \Gamma(1 + \frac{n}{2})} - 1 = \frac{\Gamma(\frac{n}{2}) (1 + \frac{n}{2}) \Gamma(1 + \frac{n}{2})}{\Gamma(1 + \frac{n}{2}) \frac{n}{2} \Gamma(\frac{n}{2})} - 1$$

$$= \frac{1 + \frac{n}{2}}{\frac{n}{2}} - \frac{\frac{n}{2}}{\frac{n}{2}} = \frac{2}{n} \text{ and } V\, T_1(\underline{X}) = \frac{2}{n} \sigma^4 = \text{FCRLB}.$$