15] Warning! If Y is not from an exponential family, generally $I_1(\theta) \neq -E\left[\frac{d^2}{d\theta^2}\log f(Y|\theta)\right]$

16] Know If $\frac{d}{d\theta}\int f(y|\theta)\,dy = \int \frac{d}{d\theta} f(y|\theta)\,dy$ and if $Y_1, \ldots, Y_n$ are iid, then $I_n(\theta) = n\,I_1(\theta)$.
These conditions hold for 1 parameter exp fam's by remark 3.3. Replace integrals by sums for pmfs.

proof] $I_n(\theta) = E_\theta\left[\left(\frac{d}{d\theta}\log f(\underset{\sim}{Y}|\theta)\right)^2\right] = E_\theta\left[\left(\frac{d}{d\theta}\log \prod f(Y_i|\theta)\right)^2\right]$

$$= E_\theta\left[\left(\sum_{i=1}^n \frac{d}{d\theta}\log f(Y_i|\theta)\right)^2\right] = \underbrace{\sum_{i=1}^n E\left[\left(\frac{d}{d\theta}\log f(Y_i|\theta)\right)^2\right]}_{n I_1(\theta)}$$

$$+ \sum_{i}\sum_{j \atop i\neq j} E_\theta\left[\frac{d}{d\theta}\log f(Y_i|\theta)\,\frac{d}{d\theta}\log f(Y_j|\theta)\right] = n\,I_1(\theta) + 0$$

since last summand $= E_\theta\left[\frac{d}{d\theta}\log f(Y_i|\theta)\right] E_\theta\left[\frac{d}{d\theta}\log f(Y_j|\theta)\right]$

and $E_\theta\left[\frac{d}{d\theta}\log f(Y_i|\theta)\right] = E_\theta\left[\frac{\frac{d}{d\theta}f(Y|\theta)}{f(Y|\theta)}\right] = \int_y \frac{\frac{d}{d\theta}f(y|\theta)}{f(y|\theta)} f(y|\theta)\,dy$

$$= \int_y \frac{d}{d\theta} f(y|\theta)\,dy = \frac{d}{d\theta}\int_y f(y|\theta)\,dy = \frac{d}{d\theta}1 = 0.$$

17) Know The Information inequality usually does not hold if the family is not an exp fam. So for these families, $V_\theta(W) < FCRLB_n(\tau(\theta))$ often occurs.

ex] Let $Y_1, \ldots, Y_n$ be iid $U(0,\theta)$.

$f(y|\theta) = \frac{1}{\theta} I(0<y<\theta)$

$\log f(y|\theta) = -\log(\theta) \quad I(0<y<\theta) \qquad$ on y

$\frac{d}{d\theta}\log f(y|\theta) = \frac{-1}{\theta}$

So $I_1(\theta) = E_\theta\left[\left(\frac{d}{d\theta} \log f(Y|\theta)\right)^2\right] = E \frac{1}{\theta^2} = \frac{1}{\theta^2}$.

So "$FCRLB_n(\theta)$" $= \dfrac{[\tau'(\theta)]^2}{n I_1(\theta)} = \dfrac{1}{n \frac{1}{\theta^2}} = \dfrac{\theta^2}{n}$.

Can show $E_\theta\left[\frac{n+1}{n} Y_{(n)}\right] = \theta$ but

$V_\theta\left[\frac{n+1}{n} Y_{(n)}\right] = \dfrac{\theta^2}{n(n+2)} < \dfrac{\theta^2}{n}$.

Information inequality did not hold, but the uniform dist is not an exp fam.

p166

18] * Suppose $Y_1, \ldots, Y_n$ are iid from a 1PREF. If $W(\underline{Y})$ is an unbiased estimator of $\tau(\theta)$ and $V_\theta(W(\underline{Y})) = FCRLB_n(\tau(\theta)) = \dfrac{[\tau'(\theta)]^2}{n I_1(\theta)}$ $\forall \theta \in \Theta$,

then $W(\underline{Y})$ is the UMVUE of $\tau(\theta)$.

p172

19] Rule of thumb (can't be used for proofs):
Suppose $Y_1, \ldots, Y_n$ are iid from a 1P REF with complete suff stat $T = \sum_{i=1}^n t(Y_i)$.
Let $W(\underline{Y}) = g(T(\underline{Y}))$ be an unbiased estimator of $\tau(\theta) = E_\theta[g(T(\underline{Y}))]$. Then $V_\theta(g(T(\underline{Y}))) > \dfrac{[\tau'(\theta)]^2}{n I_1(\theta)} = FCRLB_n(\tau(\theta))$

$\forall \theta$ if g is nonlinear.

skip

$V_\theta[g(T(\underline{Y}))] = \dfrac{[\tau'(\theta)]^2}{n I_1(\theta)}$ $\forall \theta$ only if

$g(T(\underline{Y})) = a + b\,T(\underline{Y})$ for some constants a and b.

Rule of thumb: For 1PREFs, FCRLB is achieved if $\tau(\theta) = E_\theta(a + b\,T(\underline{Y}))$ but not if $\tau(\theta) = E_\theta[g(T(\underline{Y}))]$ where ex] $Y_1, \ldots, Y_n$ iid $N(\mu, 1)$, $T(\underline{Y}) = \sum_{i=1}^{n} Y_i$.

FCRLB is achieved for $\tau(\theta) = \mu = E_\theta(\frac{1}{n} \sum Y_i)$

but not for $\tau_g(\mu) = E_\theta\left[\left(\frac{\sum Y_i}{n}\right)^2\right]$. Note that $\left(\frac{\sum Y_i}{n}\right)^2$ is the UMVUE of $\tau_g(\mu)$ by LSU

since $T(\underline{Y})$ is complete.

20] Know The 5th major question on E3, final & qual is finding the UMVUE $U = g(T)$ for $\tau(\theta)$ finding $I_1(\theta)$, $I_1(\tau(\theta))$, $FCRLB_n(\tau(\theta)) = \frac{[\tau'(\theta)]^2}{n I_1(\theta)}$

and determining whether $V_\theta(g(T)) \stackrel{?}{=} \frac{[\tau'(\theta)]^2}{n I_1(\theta)}$,

ie does $V_\theta(g(T))$ achieve the FCRLB.

ex] see ex 6.4, 6.5, 6.6, 6.7

21] Know P160 LSU gives 2 ways to find the UMVUE of $\tau(\theta)$ given a complete suff stat $T(\underline{Y})$.

i) Guess g and show $E_\theta(g(T(\underline{Y}))) = \tau(\theta)$ $\forall \theta \in \Theta$

ii) Find any unbiased estimator $W(\underline{Y})$ of $\tau(\theta)$. Then $U = g(T(\underline{Y})) = \underbrace{E[W(\underline{Y}) \mid T(\underline{Y})]}_{\text{simplify for full credit}}$ is the UMVUE of $\tau(\theta)$.

Tip: $g(T) = aT^2 + bT + c$ is often a good guess.

ex] Let $Y_1, \ldots, Y_n$ be iid $U(0, \theta)$, $\theta > 0$. Then $T(\underline{Y}) = Y_{(n)}$ is complete and $E_\theta(Y_{(n)}) = \frac{n}{n+1}\theta$. Find the UMVUE of θ.

soln $\frac{n+1}{n} Y_{(n)}$ by LSU since $E_\theta\left[\frac{n+1}{n} Y_{(n)}\right] = \theta$ $\forall \theta > 0$.

ex] Let $X_1, \dots, X_n$ be iid Poisson(λ).
want to estimate $\tau(\lambda) = e^{-\lambda} = P(X_1 = 0)$.
$T(\underline{X}) = \sum_{i=1}^{n} X_i$ is complete.
Let $W(\underline{X}) = I(X_1 = 0)$ so $E_\lambda(W(\underline{X})) = P_\lambda(X_1 = 0) = e^{-\lambda}$.
Now $E_\lambda(W(\underline{X}) \mid \Sigma X_i = u) = P_\lambda[X_1 = 0 \mid \sum_{i=1}^{n} X_i = u]$

$$= \frac{P_\lambda(X_1 = 0, \sum_{i=1}^{n} X_i = u)}{P_\lambda(\sum_{i=1}^{n} X_i = u)} = \frac{P_\lambda(X_1=0)\ \underbrace{P_\lambda(\sum_{i=2}^{n} X_i = u)}_{\text{Pois}[(n-1)\lambda]}}{\underbrace{P_\lambda(\sum_{i=1}^{n} X_i = u)}_{\text{pois}[n\lambda]}}$$

$$= \frac{e^{-\lambda}\ e^{-(n-1)\lambda}[(n-1)\lambda]^u}{u!} \ \frac{u!}{e^{-n\lambda}(n\lambda)^u} = \left(\frac{n-1}{n}\right)^u$$

So the UMVUE is $\left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i}$
since $E[W \mid \Sigma X_i]$ is the UMVUE by LSU.

Note: This type of problem is very hard.

Tips p161: i) If $Y_1, \dots, Y_n$ are iid Pois(θ), $Y_1 \mid \sum_{i=1}^{n} Y_i = x \sim \text{bin}(x, \frac{1}{n})$.
ii) If $Y_1, \dots, Y_n$ are iid Ber(ρ), $(Y_1 \mid \sum_{i=1}^{n} Y_i = x) \sim \text{ber}(\frac{x}{n})$.
iii) If $Y_1, \dots, Y_n$ are iid $N(\mu, \sigma^2)$, $(Y_1 \mid \sum_{i=1}^{n} Y_i = x) \sim N(\frac{x}{n}, \sigma^2(1 - \frac{1}{n}))$
iv) If $Y_1, \dots, Y_n$ iid, then $E(Y_1 \mid \sum_{i=1}^{n} Y_i = x) = \frac{x}{n}$.
see HW 10 #1 = 6.7 and problems 6.10 and 6.12.

ex] $X_1, \dots, X_n$ are iid Ber(ρ). Find the UMVUE of $\tau(\rho) = \rho(1-\rho)$.
Know $\bar{X}$ is complete and $E\bar{X} = \rho$.

method i) Try $a(\bar{X})^2 + b(\bar{X}) + c$; get $a = \frac{-n}{n-1}$, $b = \frac{n}{n-1}$, $c = 0$
method ii) Try $U = d\,\bar{X}(1-\bar{X})$.
$E_\rho(U) = d\ E_\rho(\bar{X} - (\bar{X})^2) = d\ E_\rho\left[\frac{1}{n}\Sigma X_i - \frac{1}{n^2}(\Sigma X_i)^2\right]$

$$= d\left[\rho - \frac{1}{n^2}\left(V(\Sigma X_i) + (E(\Sigma X_i))^2\right)\right]$$