$$= d\left[\rho - \frac{1}{n^2}\left(n\rho(1-\rho) + n^2\rho^2\right)\right] = d\left[\rho - \frac{\rho}{n} + \frac{\rho^2}{n} - \rho^2\right]$$

$\sum X_i \sim \text{bin}(n,\rho)$

$$= d\left[\frac{n-1}{n}\rho + \rho^2\frac{1-n}{n}\right] = d\,\frac{n-1}{n}(\rho-\rho^2) = d\,\frac{n-1}{n}\rho(1-\rho)$$

So take $d = \frac{n}{n-1}$ and $U = \frac{n}{n-1}\bar{X}(1-\bar{X})$ is the UMVUE.

method iii) $W(\underline{X}) = X_1(1-X_2)$ has $E(W(\underline{X})) = EX_1 - EX_1EX_2$
$= \rho - \rho^2 = \rho(1-\rho)$. So

$$E_\rho\left[X_1(1-X_2) \mid \sum X_i = u\right] = P_\rho\left[X_1(1-X_2)=1 \mid \sum_{i=1}^n X_i = u\right]$$

trick: $X_1(1-X_2) \in \{0,1\}$, so Bernoulli, $X_1(1-X_2)=1$ iff $X_1=1, X_2=0$

$E(Y) = P(Y=1)$ if $Y$ is Ber

$$= P_\rho\Big(X_1=1, X_2=0, \underbrace{\sum_{i=3}^n X_i = u-1}_{\text{bin}(n-2,\rho)}\Big) \Big/ P_\rho\Big(\underbrace{\sum_{i=1}^n X_i = u}_{\text{bin}(n,\rho)}\Big)$$

$$= \frac{\overbrace{\rho(1-\rho)}^{P(X_1=1)P(X_2=0)}\binom{n-2}{u-1}\rho^{u-1}(1-\rho)^{n-2-u+1}}{\binom{n}{u}\rho^u(1-\rho)^{n-u}} = \frac{\binom{n-2}{u-1}}{\binom{n}{u}} = \frac{\left[\frac{(n-2)!}{(u-1)!(n-1-u)!}\right]}{\left[\frac{n!}{u!(n-u)!}\right]}$$

$$= \frac{(n-2)!\,u(u-1)!\,(n-u)(n-1-u)!}{n(n-1)(n-2)!\,(u-1)!\,(n-1-u)!} = \frac{u(n-u)}{n(n-1)}$$

So the UMVUE is $U(\underline{X}) = \dfrac{\sum X_i\,(n - \sum X_i)}{n(n-1)} = \dfrac{n}{n-1}\bar{X}(1-\bar{X})$.

Ch7 1] p183 A *hypothesis* is a statement about a population parameter.

2] p. 183 In hypothesis testing there are 2 competing hypotheses: the *null hypothesis* $H_0$ and the *alternative hypothesis* $H_1 \equiv H_A$.

3) Let $\theta \in \Theta$. $H_0: \theta \in \Theta_0$ $H_1$ $\theta \in \Theta_1$

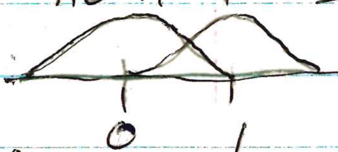
$\Theta_0 \cap \Theta_1 = \phi$, $\Theta_0, \Theta_1 \subseteq \Theta$.

4] p. 183 A hypothesis test is a rule

for rejecting $H_0$.

ex] p 184 Often there is a rejection region $R$.
Reject $H_0$ if test statistic $T(\underline{y}) \in R$, otherwise fail to reject $H_0$.

ex] $Y \sim N(\mu, \frac{1}{9})$, $\mu \in \{0, 1\}$. $\sigma^2 = \frac{1}{9}$, $3\sigma = 1$
$H_0$ $\mu = 0$ $H_1$ $\mu = 1$ Reject $H_0$ if $Y \geq \frac{1}{2}$
otherwise fail to reject $H_0$.




5] p. 183

| true hypothesis | Decision: reject $H_0$ | fail to reject $H_0$ |
|---|---|---|
| $H_0$ | type I error | |
| $H_1$ | | type II error |

A type I error is rejecting $H_0$ when $H_0$ is true
A type II error is failing to reject $H_0$ when $H_0$ is false.

6] Know P. 183 The power function of a hypothesis test is $\beta(\underline{\theta}) = P_{\underline{\theta}}(H_0 \text{ is rejected})$ for $\underline{\theta} \in \Theta$.

Want $\beta(\underline{\theta}) \simeq 0$ for $\underline{\theta} \in \Theta_0$ and $\beta(\underline{\theta}) \simeq 1$ for $\underline{\theta} \in \Theta_1$.

ex] $\beta(\theta) = P_\theta(T(\underline{Y}) \in R)$ if there is a test statistic $T(\underline{Y})$ with rejection region $R$.

Fact] For $\theta \in \Theta_0$, $\beta(\theta) = P_\theta(\text{type I error})$. For $\theta \in \Theta_1$ $\beta(\theta) = 1 - P_\theta(\text{type II error})$.

ex] $Y \sim N(\mu, \frac{1}{9})$ $\mu \in \{0, 1\} = \Theta$, $\Theta_0 = \{0\}$, $\Theta_1 = \{1\}$

$\beta(0) = P_0(Y > \frac{1}{2}) \underset{\substack{\uparrow \\ Y \sim N(0, \frac{1}{9})}}{=} P\left(\frac{Y-0}{\frac{1}{3}} > \frac{\frac{1}{2}}{\frac{1}{3}}\right) = P(Z > 1.5)$

$= 1 - P(\underset{\substack{\uparrow \\ N(0,1)}}{Z} < 1.5) \underset{\substack{\uparrow \\ \text{table}}}{=} 1 - .9332 = .0668$

$\beta(1) = P_1(Y > \frac{1}{2}) \underset{\substack{\uparrow \\ Y \sim N(1, \frac{1}{9})}}{=} P\left(\frac{Y-1}{1/3} > \frac{\frac{1}{2}-1}{1/3}\right) = P(Z > -1.5) = .9332$

7) Know p184 For $0 \le \alpha \le 1$, a test with power function $\beta(\theta)$ is a size $\alpha$ test if $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$ and a level $\alpha$ test if $\sup_{\theta \in \Theta_0} \beta(\theta) \le \alpha$.

Tip! Often $\Theta_0 = (a, \theta_0]$, $\Theta_1 = (\theta_0, b)$ and $\alpha = \beta(\theta_0) = P_{\theta_0}(\text{type I error})$.

ex] The last ex is a level 0.07 test and a size .0668 test.

p185

§7.2 8] Consider all level $\alpha$ tests of $H_0\ \theta \in \Theta_0$ vs $H_1\ \theta \in \Theta_1$. A uniformly most powerful (UMP) level $\alpha$ test is a level $\alpha$ test with power function $\beta_{UMP}(\theta) \ge \beta(\theta)\ \forall\, \theta \in \Theta_1$ where $\beta$ is the power function for any level $\alpha$ test of $H_0$ vs $H_1$.

9) Know Neyman Pearson NP Lemma: Consider testing $H_0\ \theta = \theta_0$ vs $H_1\ \theta = \theta_1$ where the pdf or pmf

corresponding to $\theta_i$ is $f(\underline{y}|\theta_i)$ for $i = 0, 1$.
The level $\alpha$ UMP test
rejects $H_0$ if $f(\underline{y}|\theta_1) > k\, f(\underline{y}|\theta_0)$

and rejects $H_0$ with prob $\gamma$ if $f(\underline{y}|\theta_1) = k\, f(\underline{y}|\theta_0)$
for some $k \geq 0$.

$$\alpha = \beta(\theta_0) = P_{\theta_0}[f(\underline{Y}|\theta_1) > k\, f(\underline{Y}|\theta_0)] + \gamma\, P_{\theta_0}[f(\underline{Y}|\theta_1) = k\, f(\underline{Y}|\theta_0)].$$

Note] With pdfs, $\gamma = 0$ and the rejection region $\underline{y} \in R$ is $f(\underline{y}|\theta_1) > k\, f(\underline{y}|\theta_0)$.

10] Know There is a $k$ where you reject $H_0$ if $\dfrac{f(\underline{y}|\theta_1)}{f(\underline{y}|\theta_0)} > k$. Put the test into "useful form" by simplifying so that you reject $H_0$ if $T(\underline{y}) > d$ (if the ratio is an increasing function of $T(\underline{y})$) or if $T(\underline{y}) < c$ (if ratio is a decreasing function of $T(\underline{y})$)

11] p186 Assume $\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0)$

Know Test $H_0$ $\theta \leq \theta_0$ vs $H_1$ $\theta > \theta_0$
or $H_0$ $\theta < \theta_0$ vs $H_1$ $\theta \geq \theta_0$ } right tail
or $H_0$ $\theta \geq \theta_0$ vs $H_1$ $\theta < \theta_0$
or $H_0$ $\theta > \theta_0$ vs $H_1$ $\theta \leq \theta_0$. } left tail.

Pick $\theta_1 \in \Theta_1$ and use NP to find the UMP level $\alpha$ test for $H_0^*$ $\theta = \theta_0$ vs $H_1^*$ $\theta = \theta_1$.
So reject $H_0^*$ if $f(\underline{y}|\theta_1) > k\, f(\underline{y}|\theta_0)$ and reject $H_0^*$ with prob $\gamma$ if $f(\underline{y}|\theta_1) = k\, f(\underline{y}|\theta_0)$ for some $k \geq 0$ where $\alpha = \beta(\theta_0)$ in 9]. This test

is the UMP level $\alpha$ test for $H_0\ \theta \in \Theta_0$ vs $H_1\ \theta \in \Theta_1$ if $k$ does not depend on the value of $\theta_1 \in \Theta_1$.
(So $k$ depends only on $\theta_0$ and $\alpha$.)

12] Know p. 186 one sided tests for exp families.
Let $Y_1, \ldots, Y_n$ be iid with pdf or pmf

$$f(y|\theta) = h(y)\, c(\theta) \exp[w(\theta)\, t(y)]$$

where $w(\theta)$ is increasing (eg $w'(\theta) > 0$).
Let $T(\underline{y}) = \sum_{i=1}^{n} t(y_i)$.

I) Let $\theta_1 > \theta_0$

I > goes with $H_1$ & $\theta_1 > \theta_0$

Consider
$H_0\ \theta = \theta_0$ vs $H_1\ \theta = \theta_1$
$H_0\ \theta = \theta_0$ vs $H_1\ \theta > \theta_0$
$H_0\ \theta \le \theta_0$ vs $H_1\ \theta > \theta_0$.

The UMP level $\alpha$ test rejects $H_0$ if $T(\underline{y}) > k$ and rejects $H_0$ with prob $\gamma$ if $T(\underline{y}) = k$ where $\alpha = P_{\theta_0}(T(\underline{Y}) > k) + \gamma\, P_{\theta_0}(T(\underline{Y}) = k)$.

II Let $\theta_1 < \theta_0$.

II < goes with $H_1$ & $\theta_1 < \theta_0$

Consider
$H_0\ \theta = \theta_0$ vs $H_1\ \theta = \theta_1$
$H_0\ \theta = \theta_0$ vs $H_1\ \theta < \theta_0$
$H_0\ \theta \ge \theta_0$ vs $H_1\ \theta < \theta_0$.

The UMP level $\alpha$ test rejects $H_0$ if $T(\underline{y}) < k$ and rejects $H_0$ wp $\gamma$ if $T(\underline{y}) = k$ where $\alpha = P_{\theta_0}(T(\underline{Y}) < k) + \gamma\, P_{\theta_0}(T(\underline{Y}) = k)$.

13] Mnemonic p. 199 The ">" or "<" goes with $H_1$ and "$\theta_1 > \theta_0$" or "$\theta_1 < \theta_0$"
The inequality in the rejection region is the same as the inequality in the alternative hypothesis.

14] Know suppose you find

$f(y|\theta) = h(y)\, c(\theta) \exp[\tilde{w}(\theta)\, \tilde{t}(y)]$

where $\tilde{w}$ is decreasing (eg $\tilde{w}'(\theta) < 0$).

Then set $w(\theta) = -\tilde{w}(\theta)$ and

$t(y) = -\tilde{t}(y)$.

ex] See ex 7.2 – 7.5. See problems 7.10, 7.13, 7.14, 7.15, and 7.16 solns in ch 12.

15] 6th problem for E3 final qual is finding UMP tests via NP lemma or exp fam theory.

ex] $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$, $\sigma^2 > 0$ known.

Find UMP level $\alpha$ test for $H_0\ \mu = \mu_0$ vs $H_1\ \mu > \mu_0$.

soln method I) $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)}$

omit

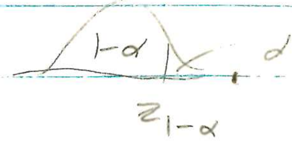
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\, e^{-\mu^2/2\sigma^2} \exp\Big[\underbrace{\frac{\mu}{\sigma^2}}_{w(\mu)}\ \underbrace{x}_{t(x)}\Big]$$

$w'(\mu) = \frac{1}{\sigma^2} > 0$ so $w \uparrow$. Let $P(Z \le z_\alpha) = \alpha$ if $Z \sim N(0,1)$

So reject $H_0$ if $\sum_{i=1}^n X_i > k$ where

$\alpha = P_{\mu_0}(\sum X_i > k) \underset{\bar{X} \overset{H_0}{\sim} N(\mu_0, \frac{\sigma^2}{n})}{=} P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}\right)$. So UMP

test rejects $H_0$ if $\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}$.



Method II) Use NP lemma: Fix $\mu_1 > \mu_0$.

$$\frac{f(\underset{\sim}{x}|\mu_1)}{f(\underset{\sim}{x}|\mu_0)} = \frac{e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu_1)^2}}{e^{-\frac{1}{2\sigma^2}\Sigma(x_i-\mu_0)^2}} > k \quad \text{iff}$$

$$\frac{e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}\; e^{\frac{-n\mu_1^2}{2\sigma^2}} \cdot e^{\frac{\mu_1}{\sigma^2}\Sigma x_i}}{\underbrace{e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}\; e^{\frac{-n\mu_0^2}{2\sigma^2}}}_{\text{positive constant}}\; e^{\frac{\mu_0}{\sigma^2}\Sigma x_i}} > k$$

iff $\exp\left(\frac{1}{\sigma^2}\underbrace{(\mu_1-\mu_0)}_{>0 \text{ if } \mu_1>\mu_0}\Sigma X_i\right) > k'$ ← key step

iff $\Sigma X_i > k''$ where $P_{\mu_0}(\Sigma X_i > k'') = \alpha$

key step → Now $k''$ only depends on $\mu_0$ and $\alpha$ not on $\mu_1 > \mu_0$. So the UMP test rejects $H_0$ if $\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}$.

ex] $X_1, \ldots, X_n$ iid Ber($\rho$), $\rho \in (0,1)$. $f(y) = \rho^y (1-\rho)^{1-y}$ $y \in \{0,1\}$
Find UMP level $\alpha$ test for $H_0$ $\rho = \rho_0$ vs $H_A$ $\rho > \rho_0$.

Soln (omit) $\left( f(x) = (1-\rho)\exp\left[\underbrace{\log\left(\frac{\rho}{1-\rho}\right)}_{w(\rho)\uparrow} x\right] \text{ so reject } H_0 \text{ if } \Sigma x_i > k \text{ and wp } \gamma \text{ if } \Sigma x_i = k \right)$

Fix $\rho_1 > \rho_0$. Now $\frac{f(\underset{\sim}{x}|\rho_1)}{f(\underset{\sim}{x}|\rho_0)} = \frac{\rho_1^{\Sigma x_i}(1-\rho_1)^{n-\Sigma x_i}}{\rho_0^{\Sigma x_i}(1-\rho_0)^{n-\Sigma x_i}}$

$= \left(\frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1)}\right)^{\Sigma x_i}\left(\frac{1-\rho_1}{1-\rho_0}\right)^n > k$ iff $\Sigma x_i \log\left(\underbrace{\frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1)}}_{\substack{\rho_1>\rho_0 \\ 1-\rho_0>1-\rho_1}}\right) > k'$

iff $\Sigma x_i > k''$. So reject $H_0$ if $\Sigma x_i > c$

If $\rho_1 > \rho_0$ this test depends only on $\alpha$ and $\rho_0$.
So the UMP level $\alpha$ test rejects $H_0$ if $\sum_{i=1}^n x_i > c$
and rejects $H_0$ wp $\gamma$ if $\sum_{i=1}^n x_i = c$

(*) where $\alpha = P_{\rho_0}\left(\sum_{i=1}^n X_i > c\right) + \gamma \, P_{\rho_0}\left(\sum_{i=1}^n X_i = c\right)$.

Suppose $\rho_0 = \frac{1}{2}$ and $n = 8$. Then $\sum_{i=1}^n X_i \overset{H_0}{\sim} \text{bin}(8, \frac{1}{2})$

| $c$ | 8 | 7 | 6 |
|---|---|---|---|
| $P_{\frac{1}{2}}\left(\sum_{i=1}^8 X_i = c\right)$ | $\binom{8}{8}\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^8$ | $\binom{8}{7}\left(\frac{1}{2}\right)^8 = 8\left(\frac{1}{2}\right)^8$ | $\frac{8 \cdot 7}{2}\left(\frac{1}{2}\right)^8 =$ |
| | $p_8 = 0.0039063$ | $p_7 = 0.03125$ | $p_6$ $0.10937$ |

If $\alpha = .05$, reject $H_0$ if $\sum x_i > 6$ and reject $H_0$ wp $\gamma$ if $\sum x_i = \boxed{6 = c}$ where by (*)

$$\alpha = p_8 + p_7 + \gamma p_6 \quad \text{or} \quad \gamma = \frac{\alpha - p_7 - p_8}{p_6}$$

$$= \frac{.05 - .0039063 - .03125}{.10937} = \frac{.0148437}{.10937} \approx .13572 = \gamma$$

p 188 In general, (*) gives $\gamma = \dfrac{\alpha - P_{\rho_0}(\sum x_i > c)}{P_{\rho_0}(\sum x_i = c)}$.

ex] $X_1, \ldots, X_n$ iid $f(x|\theta) = \dfrac{\theta}{x^{\theta+1}}$, $x > 1$.

a) Find UMP level $\alpha$ = size $\alpha$ test of $H_0$ $\theta = 1$ vs $H_1$ $\theta > 1$

soln I) $f(x|\theta) = I(x>1)\, \theta \, \exp[\underbrace{-(\theta+1)}_{\tilde{w}(\theta)\downarrow} \log x]$

$$= I(x>1)\, \theta \, \exp[\underbrace{(\theta+1)}_{w(\theta)\uparrow} \underbrace{[-\log(x)]}_{t(x)}]$$

$T(\underline{X}) = -\sum_{i=1}^n \log(X_i)$. UMP test rejects $H_0$