if $-\sum_{i=1}^{n} \log x_i > k$ where

$\alpha = P_1\left(-\sum_{i=1}^{n} \log X_i > k\right)$.

II) Let $\theta_1 > 1$. Now $f(\underline{x}|\theta) = \dfrac{\theta^n}{\prod x_i^{\theta+1}} I(x_{(1)} > 1)$

and $\dfrac{f(\underline{x}|\theta_1)}{f(\underline{x}|1)} = \dfrac{\theta_1^n}{(\prod x_i)^{\theta_1+1}} \dfrac{(\prod x_i)^2}{1^n} = \dfrac{\theta_1^n}{(\prod x_i)^{\theta_1-1}} > k$

if $-\underbrace{(\theta_1-1)}_{>0 \text{ if } \theta_1>1} \sum \log x_i > k'$

if $-\sum \log x_i > k''$ where $P_1(-\sum \log X_i > k'') = \alpha$

Above test is UMP by NP since test did not depend on $\theta_1 > 1$.

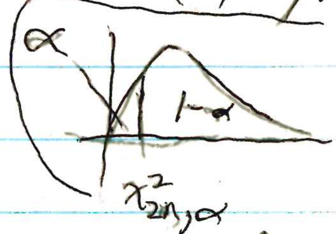
b) Implement the test using a χ^2 table.

Sol'n] $f(x|1) = \dfrac{1}{x^2}$, $x > 1$. For $t > 0$,

$P(\log X \le t) = P(X \le e^t) = \int_1^{e^t} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_1^{e^t}$
$= 1 - e^{-t}$. So $\log X \sim EXP(1) \sim G(1,1)$.

So $\sum_{i=1}^{n} \log(X_i) \sim G(n,1)$ and $2\sum_{i=1}^{n} \log(X_i) \sim \chi^2_{2n} \sim G(n,2)$.

So reject Ho if $-\sum \log x_i > c \Leftrightarrow$
reject Ho if $2\sum \log x_i < \chi^2_{2n,\alpha}$



§7.3 16) Know p. 192 Let $Y_1, \ldots, Y_n$ have pdf or pmf $f(\underline{y}|\underline{\theta})$ for $\underline{\theta} \in \Theta$. Let $\hat{\underline{\theta}}$ be the MLE of $\underline{\theta}$ and let $\hat{\underline{\theta}}_0$ be the MLE of

$\underset{\sim}{\theta}$ if $\underset{\sim}{\theta} \in \Theta_0 \subseteq \Theta$. A likelihood ratio test (LRT) test statistic for testing $H_0\ \underset{\sim}{\theta} \in \Theta_0$ vs $H_1\ \underset{\sim}{\theta} \in \Theta_0^c$ is

$$\lambda(\underset{\sim}{y}) = \frac{L(\hat{\underset{\sim}{\theta}}_0 \mid \underset{\sim}{y})}{L(\hat{\underset{\sim}{\theta}} \mid \underset{\sim}{y})}.$$

The likelihood ratio test (LRT) rejects H_0 if $\lambda(\underset{\sim}{y}) \le c$ where $0 \le c \le 1$ and $\alpha = \sup_{\underset{\sim}{\theta} \in \Theta_0} P_{\underset{\sim}{\theta}}(\lambda(\underset{\sim}{Y}) \le c)$. The rejection region $R = \{\underset{\sim}{y} \mid \lambda(\underset{\sim}{y}) \le c\}$.

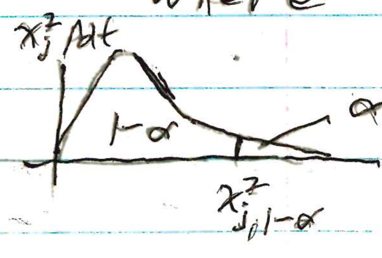
17] If $\Theta = (a,b)$, $\Theta_0 = (a, \theta_0]$ and $\Theta_1 = \Theta_0^c = (\theta_0, b)$, then typically $\alpha = \beta(\theta_0) = P_{\theta_0}(\lambda(\underset{\sim}{Y}) \le c)$.

ex] $\Theta_0 = \underbrace{\{\theta_0\}}_{\text{singleton set}}$. Then $\hat{\theta}_0 = \theta_0$.
eg $\Theta_0 = \{0\} \Rightarrow \hat{\theta}_0 = 0$.

Tip] Try to write $\lambda(\underset{\sim}{y})$ as a function of a min suff stat $T(\underset{\sim}{y})$.

18] Know p. 193 Rule of thumb: Under strong regularity conditions $-2\log\lambda(\underset{\sim}{Y}) \approx \chi^2_j$ for large n where $j = r - q$, r = # of free parameters specified by $\underset{\sim}{\theta} \in \Theta$ and q = # free parameters specified by $\underset{\sim}{\theta} \in \Theta_0$. So an approximate LRT rejects H_0 if $-2\log\lambda(\underset{\sim}{y}) > \chi^2_{j,1-\alpha}$ where

$P(X \le \chi^2_{j,\alpha}) = \alpha$ if $X \sim \chi^2_j$.



19) Know 7th type of E3 final qual problem is find LRT for H_0 $\underline{\theta} \in \Theta_0$ vs H_1 $\underline{\theta} \in \Theta_0^c$ possibly simplifying $\lambda(\underline{y}) \le c$ to $T(\underline{y}) \le k$ or $T(\underline{y}) \ge k$ or using point 18].

ex] Suppose $\sum_{j=1}^{5} \rho_j = 1$

H_0 $\underbrace{\rho_1 = \rho_2 = \rho_3}$ and $\underbrace{\rho_4 = \rho_5}$ | H_1 not H_0

$\rho_1 + \rho_1 + \rho_1 + \rho_4 + \rho_4 = 1$

$\rho_4 = \frac{1 - 3\rho_1}{2}$ so $g = 1$

$r = 4$ since $\rho_5 = 1 - (\rho_1 + \rho_2 + \rho_3 + \rho_4)$

so $j = r - g = 3$

ex] a) $\Theta_0 = \{\theta_0\}$ $g = 0$, b) H_0 $\mu = 100$ H_A $\mu \ne 100$ $r = 1$ $g = 0$

ex] See ex 7.8, 7.9, 7.10, 7.11

ex] Problem 7.11 $X_1, \dots, X_n$ iid $N(\mu, \sigma^2)$, σ^2 known

H_0 $\mu = \mu_0$ vs H_1 $\mu \ne \mu_0$

a) Find the LRT.

Soln $\hat{\mu}_0 = \mu_0$ $\hat{\mu} = \bar{x}$

$$L(\mu) = f(\underline{x} \mid \mu, \sigma^2) = \prod \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2} = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left[\frac{-1}{2\sigma^2}\sum (x_i - \mu)^2\right]$$

$$\text{so } \lambda(\underline{x}) = \frac{L(\mu_0)}{L(\hat{\mu})} = \frac{\exp\left[\frac{-1}{2\sigma^2}\sum (x_i - \mu_0)^2\right]}{\exp\left[\frac{-1}{2\sigma^2}\sum (x_i - \bar{x})^2\right]}$$

The α level LRT rejects H_0 if $\lambda(\underline{x}) \le c$ where $P_{\mu_0}(\lambda(\underline{X}) \le c) = \alpha$.

b) Show $-2 \log \lambda(\underline{x})$ is a function of $\bar{x} - \mu_0$

Soln $\log \lambda(\underline{x}) = \frac{-1}{2\sigma^2}\left[\Sigma(x_i-\mu_0)^2 - \Sigma(x_i-\bar{x})^2\right]$

$$= \frac{-1}{2\sigma^2}\left[\Sigma x_i^2 - 2\mu_0 \Sigma x_i + n\mu_0^2 - \Sigma x_i^2 + 2\bar{x}\,\Sigma x_i - n\bar{x}^2\right]$$

$$= \frac{-1}{2\sigma^2}\left[-2n\mu_0\bar{x} + 2n(\bar{x})^2 + n\mu_0^2 - n(\bar{x})^2\right]$$

$$= \frac{-1}{2\sigma^2}\left[-2n\mu_0\bar{x} + n(\bar{x})^2 + n\mu_0^2\right]$$

$$= \frac{-n}{2\sigma^2}\left[(\bar{x})^2 - 2\mu_0\bar{x} + \mu_0^2\right]$$

$$= \frac{-n}{2\sigma^2}(\bar{x}-\mu_0)^2.$$

So $-2\log\lambda(\underline{x}) = \frac{n}{\sigma^2}(\bar{x}-\mu_0)^2$.

c) Put the LRT in useful form.

Soln $-2\log\lambda(\underline{x}) = \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 \overset{H_0}{\sim} \chi^2_1$

(also $r=1$, $g=0$ since μ_0 & σ^2 are known)

Reject H_0 if $\lambda(\underline{x}) \le c$ iff $\log\lambda(\underline{x}) \le c'$

iff $-2\log\lambda \ge k$ where $P_{\mu_0}(\chi^2_1 \ge k) = \alpha$

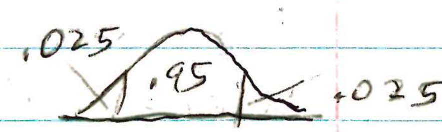
So $k = \chi^2_{1,1-\alpha}$. If $\alpha = .05$ take $k = 3.84$

χ^2 table

df	$p=\alpha$ .05
1	3.84

Note $\chi_1^2 \ge k \Leftrightarrow z > \sqrt{k}$ or $z < -\sqrt{k}$

where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$.



$\sqrt{k} = 1.96 \Leftrightarrow k = 3.8416$

	.06
-1.9	.0250
1.9	.9750

This is the usual 2 tail z test

LRT common problem 49 $\frac{3}{4}$

useful for HW11 #1

1. $X_1, \dots, X_n \overset{iid}{\sim} f(x|\theta)$

$Y_1, \dots, Y_m \overset{iid}{\sim} f(y|\mu)$

$H_1 \ \theta \neq \mu$

$H_0 \ \theta = \mu$ means $X_1, \dots, X_n, Y_1, \dots, Y_m$

are iid with $f(y/\theta)$.

so if from a $N(\mu, 1)$ family

$$\hat{\theta}_0 \left(= \hat{\mu}_0\right) = \frac{\Sigma X + \Sigma Y}{n+m}.$$

$\hat{\theta} = \bar{X}$ $\quad \hat{\mu} = \bar{Y}$

$\prod f(x_i|\theta)$ is a constant wrt μ

$\prod f(y_i|\mu)$ is a constant wrt θ

After algebra, often get an LRT test statistic like

$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\hat{\theta}_0^{\,m+n}}{\hat{\mu}^m \, \hat{\theta}^n}$$ where $\hat{\theta}_0 = \hat{\mu}_0$.

p215

Ch 8 1] Know Central Limit Theorem (CLT): Let $Y_1, \ldots, Y_n$ be iid with $E(Y_1) = \mu$ and $V(Y_1) = \sigma^2$.

Then $\sqrt{n}\,(\overline{Y}_n - \mu) \underbrace{\xrightarrow{D}}_{\approx \text{ for now}} N(0, \sigma^2)$

So $\sqrt{n}\left(\frac{\overline{Y}_n - \mu}{\sigma}\right) = \sqrt{n}\left(\frac{\sum Y_i - n\mu}{n\sigma}\right) \xrightarrow{D} N(0,1)$.

p216

2] Know 2 applications i) $\sqrt{n}\,(\overline{Y}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$

ii) Suppose $Y_n = \sum_{i=1}^{n} X_i$ where X_i are iid with mean μ_X and variance σ_X^2. Then

$\sqrt{n}\,(\overline{X} - \mu_X) = \sqrt{n}\left(\frac{Y_n}{n} - \mu_X\right) \xrightarrow{D} N(0, \sigma_X^2)$.

see 8.4 Th 2.17

ex] See ex 8.1, problem 8.4.

3] p217 know Delta Method If $g'(\theta) \neq 0$ and $\sqrt{n}\,(T_n - \theta) \xrightarrow{D} N(0, \sigma^2)$ where $\sigma^2 \equiv \sigma^2(\theta)$,

then $\sqrt{n}\,(g(T_n) - g(\theta)) \xrightarrow{D} N(0, \sigma^2 [g'(\theta)]^2)$.

Note: RHS in 1]-3] does not depend on n.

ex] See ex 8.3, 8.4, 8.5

4] If $\underbrace{W_n}_{\text{depends on } n} \xrightarrow{D} \underbrace{X}_{\text{free of } n}$, X is called the limiting distribution of W_n or asymptotic distribution.

ex] $Y_1, \ldots, Y_n$ iid $EXP(\lambda)$

a) Find the limiting distribution of $\sqrt{n}\,(\overline{Y} - \lambda)$.

Soln] $E(Y_i) = \lambda \qquad V(Y_i) = \lambda^2$

So $\sqrt{n}\,(\overline{Y} - \lambda) \xrightarrow{D} N(0, \lambda^2)$ by the CLT.

b) Find the limiting distribution of $\sqrt{n}\,((\overline{Y}_n)^2 - c)$ for appropriate constant c.

50.5

Soln] $g(\bar{Y}_n) = (\bar{Y}_n)^2$ so $c = g(\lambda) = \lambda^2$

$g'(\lambda) = 2\lambda$, $[g'(\lambda)]^2 = 4\lambda^2$. So

$\sqrt{n}\left((\bar{Y}_n)^2 - \lambda^2\right) \xrightarrow{D} N(0, \lambda^2 4\lambda^2) = N(0, 4\lambda^4)$

by the delta method.

ex] HW11 #6 = 8.3a) $\sqrt{n}\Big(\underbrace{\bar{X}_n}_{W_n} - \underbrace{\frac{3\theta}{\theta+1}}_{\tau(\theta) = EX}\Big) \xrightarrow{D} N(0, v(\theta))$

Find $E(X)$ and $V(X) \equiv v(\theta)$.

Want distribution of $\sqrt{n}\left(\frac{\bar{X}_n}{3-\bar{X}_n} - \theta\right)$.

Try $g(W_n) = g(\bar{X}_n) = \frac{\bar{X}_n}{3-\bar{X}_n}$, $g(\tau) = \frac{\tau}{3-\tau} \stackrel{?}{=} \theta$

and $\sqrt{n}\left(g(\bar{X}_n) - g(\tau)\right) \xrightarrow{D} N[0, v(\theta)(g'(\tau))^2]$

where $g'(\tau) = g'(w)\Big|_{w = \tau(\theta) = \frac{3\theta}{\theta+1}}$.

5] Know rule of thumb 8) on p244

Theorem] Let $\hat{\theta}_n$ be the MLE of θ. Then under strong regularity conditions,

$\sqrt{n}\left(\tau(\hat{\theta}_n) - \tau(\theta)\right) \xrightarrow{D} N\left(0, \frac{[\tau'(\theta)]^2}{I_1(\theta)}\right) = N[0, FCRLB_1(\tau(\theta))]$

delta method

So $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} N\left(0, \frac{1}{I_1(\theta)}\right) = N[0, FCRLB_1(\theta)]$.

6] Know Typically the delta method needs $\sqrt{n}(T_n - \theta) \xrightarrow{D} N(0, \sigma^2(\theta))$. Typically LHS comes from CLT 1] or MLE theorem 5].

p221

7] * Let $Y_1, \ldots, Y_n$ be iid from a 1PREF with