Minimal sufficient and complete statistics

MLE

Method of moments

Minimize MSE

UMVUE and FCRLB

UMP tests   NP   Exp Fam Th

LRT

large sample theory   CLT delta method

Power

exp family where $\tilde{w}(\theta)\downarrow$

consistent estimator

Show a (min suff) stat   is not complete

min suff by LSM

complete T: $\underbrace{g(T) \text{ achieves FCRLB if } g \text{ is linear but not if } g \text{ is nonlinear}}_{\text{easy}}$

1) $Y_1, \dots, Y_n$ iid

$$f(y) = \frac{1}{y} I_{[0,1]}(y) \frac{1}{\lambda} \exp\Big[\underbrace{\frac{-1}{\lambda}}_{w(\lambda)\uparrow} (-\log y)\Big] \quad \text{1PREF}$$

$\lambda > 0$. Use $t(Y) = -\log(Y) \sim EXP(\lambda) \sim \frac{\lambda}{2}\chi^2_2$.

so $-\frac{2}{\lambda}\log(Y) \sim \chi^2_2$ $\quad \left(\frac{2}{\lambda} EXP(\lambda) \sim EXP(2) \sim \chi^2_2\right)$

a) Find UMP level α test for

H_0 $\lambda = 1$ vs H_A $\lambda = 1.242$

$w(\lambda)\uparrow$, 1PREF

soln} reject H_0 if $-\sum_{i=1}^{n} \log Y_i > k$

where $P_1\left(-\sum_{i=1}^{n} \log Y_i > k\right) = \alpha$.

Find 14

b) If $n=40$, $\alpha = .01$, find the power $\beta(1.242)$.

$\sum_{i=1}^{n} t(Y_i) \sim \frac{\lambda}{2}\chi^2_{2n}$ so $\frac{2}{\lambda}\sum_{i=1}^{n} t(Y_i) \sim \chi^2_{2n}$

$$\alpha = .01 = P_1\left(-\sum_{i=1}^{40} \log Y_i > k\right) \underset{2/\lambda = 2/1 = 2}{=} P\left(-2\sum \log Y_i > 2k\right)$$

$$= P(\underset{\chi^2_{80}}{X} > 2k) \quad \text{so} \quad 2k = \chi^2_{80, .99} = 112.3$$

$$k = \frac{112.3}{2} = 56.15$$

$$\beta(1.242) = P\left(-\sum_{i=1}^{40} \log Y_i > 56.15\right) \underset{2/\lambda = \frac{2}{1.242}}{=} P\left[\frac{2}{1.242}\left(-\sum \log Y_i\right) > \underbrace{\frac{2(56.15)}{1.246}}_{90.4186}\right]$$

$$\approx P(\underset{\chi^2_{80}}{X} > 90.41) = .20.$$

$90.41 = \frac{2}{\lambda} 56.15$

$\lambda = \frac{2(56.15)}{90.41} = 1.242$

Fin rev 2.5

2) Good way to get an MSE problem.

$Y_1, \ldots, Y_n$ are iid from a 1PREF with pdf or pmf $f(y|\theta)$ and complete statistic $T(\underline{Y}) = \sum_{i=1}^n t(Y_i)$ where $t(Y_i) \sim \theta X$. Let $W_n = c\,T(\underline{Y})$ be an estimator of θ. X has a known dist (eg $X \sim G(1,1)$). So $E(X)$ and $V(X)$ are known.

a) Find $MSE(W_n)$

Th 6.5

soln] $E W_n = c \sum_{i=1}^n E\, t(Y_i) = c n \theta E(X)$

$V(W_n) = c^2 \sum_{i=1}^n V(t(Y_i)) = c^2 n \theta^2 V(X)$

$MSE(W_n) = V(W_n) + (E W_n - \theta)^2 =$

$c^2 n \theta^2 V(X) + (c n \theta E(X) - \theta)^2 = MSE(c).$

b) Find c that minimizes MSE.

soln $\dfrac{d\, MSE(c)}{dc} = 2 c n \theta^2 V(X) + 2\left(c n \theta E(X) - \theta\right) n \theta E(X) \overset{set}{=} 0$

or $c\left[n\theta^2 V(X) + n^2\theta^2 [E(X)]^2\right] = n\theta^2 E(X)$

$c = \dfrac{n\theta^2 E(X)}{n\theta^2 V(X) + n^2\theta^2 [E(X)]^2} = \boxed{\dfrac{E(X)}{V(X) + n[E(X)]^2}}$, unique

$\dfrac{d^2 MSE(c)}{dc^2} = 2\left[n\theta^2 V(X) + n^2\theta^2[E(X)]^2\right] > 0$

Note: $c \approx \dfrac{1}{nE(X)}$ for large n.

c) The UMVUE of $\theta E(X)$ is $\dfrac{T(\underline{Y})}{n}$.

So the UMVUE of θ is $\dfrac{T(\underline{Y})}{nE(X)}$ which uses $c_U = \dfrac{1}{nE(X)}$ by LSU.

(θ is the unknown parameter so $E(X)$ does not depend on θ.)

$HN(0, \sigma^2)$ $\qquad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} y_i^2$

$$L(\sigma) = c \, \frac{1}{\sigma^n} \exp\left(\frac{-1}{2\sigma^2} \sum y_i^2 \right)$$

$H_0 \; \sigma^2 = \sigma_0^2$ vs $H_1 \; \sigma^2 \neq \sigma_0^2$ LRT

$$\lambda(\underline{y}) = \frac{L(\sigma_0^2)}{L(\hat{\sigma}^2)} = \frac{\frac{1}{\sigma_0^n} \exp\left(\frac{-1}{2\sigma_0^2} \sum y_i^2 \right)}{\frac{1}{\hat{\sigma}^n} \exp\left(\frac{-1}{2\hat{\sigma}^2} \underbrace{\sum y_i^2}_{n\hat{\sigma}^2} \right)}$$

$$= \left(\frac{\hat{\sigma}}{\sigma_0} \right)^n \exp\left(\frac{-1}{2\sigma_0^2} \sum y_i^2 + \frac{n}{2} \right)$$

$$= \left(\frac{\hat{\sigma}}{\sigma_0} \right)^n \exp\left(\frac{-n\hat{\sigma}^2}{2\sigma_0^2} + \frac{n}{2} \right)$$

$X_1, \ldots, X_n$ iid

See F214 for $HN(0, \sigma_1^2)$ $\qquad Y_1, \ldots, Y_m$ iid $HN(0, \sigma_2^2)$

Final review 4

2) Let $Y_1, ..., Y_n$ be iid with pdf

$w(\phi) = \phi \uparrow \quad t(y) = -\log\left(\frac{\theta}{\theta - y}\right)$

$$f(y) = \frac{\phi}{\theta} I(0 < y < \theta) \frac{\theta}{\theta - y} \exp\left[-\phi \log\left(\frac{\theta}{\theta - y}\right)\right]$$

1PREF

for $0 < y < \theta$ where $\phi > 0$ and $\theta > 0$ **is known**. You may use the fact that

MDL

$$\log\left(\frac{\theta}{\theta - Y}\right) \sim EXP(1/\phi) \sim \frac{1}{2\phi}\chi^2_2.$$

a) Find the UMP level α test for $H_0 : \phi = 0.5$ versus $H_1 : \phi = 1$.

reject Ho if $-\sum_{i=1}^{n} \log\left(\frac{\theta}{\theta - Y}\right) > K$ where

$P_{0.5}\left[-\sum_{i=1}^{n} \log\left(\frac{\theta}{\theta - Y_i}\right) > K\right] = \alpha$.

b) Suppose $n = 10$ and $\alpha = 0.05$. Find the power $\beta(1)$ when $\phi = 1$ using the chi-square table.

too tricky

$\alpha = .05 = P_{0.5}\Big[\underbrace{-\sum_{i=1}^{n} \log \frac{\theta}{\theta - Y_i}}_{-X} > K\Big) = P_{0.5}\Big[\underbrace{\sum \log \frac{\theta}{\theta - Y}}_{X} < K'\Big)$

$= P\left(X < \underbrace{\chi^2_{20, .05}}\right)$

don't have this on a table

$K = -$ qchisq(0.05, 20) $= -10.85081$ R command

$\beta(1) = P_1\left(-\sum \log\left(\frac{\theta}{\theta - Y}\right) > -10.85081\right) = P_1\left[\sum \log\left(\frac{\theta}{\theta - Y}\right) < 10.85081\right]$

$= P\left(\frac{1}{2}\chi^2_{20} < 10.85081\right) = P\left(\chi^2_{20} < 21.7062\right) = .64308$

R command pchisq(21.70162, 20)