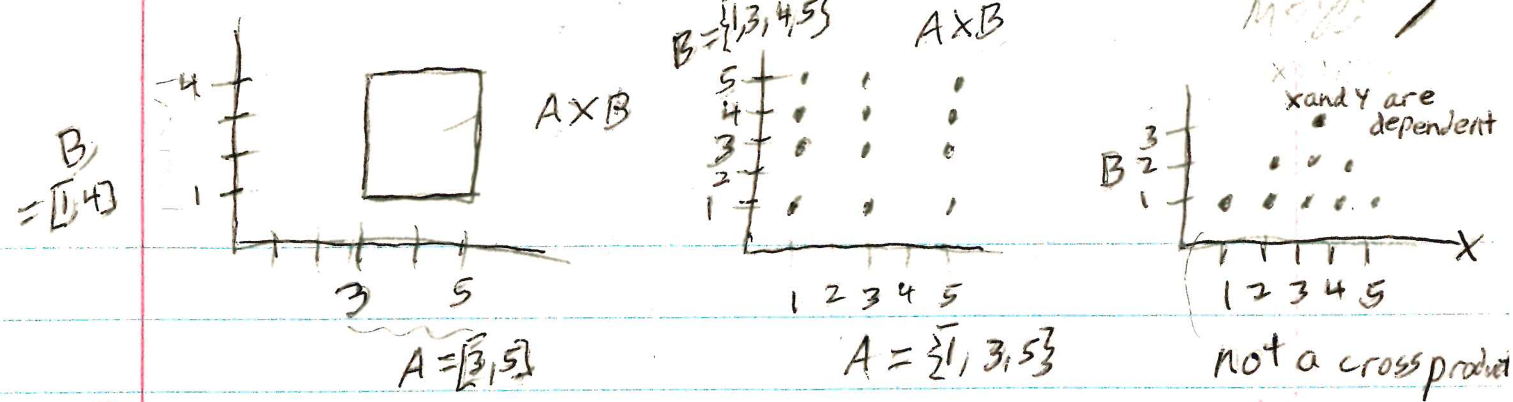


18] * p35 A necessary but not sufficient condition for independence is that the support is a cross product. Th2.2a) If the support is not a cross product, then the RVs are dependent. The RVs could be dependent or ind if the support is a cross product.

19] Th 2.2 b) If $Y_1, \dots, Y_n$ have cross product support, then they are ind iff
$f(y_1, \dots, y_n) = h_1(y_1) \cdots h_n(y_n) \quad \forall \underline{y} \in \mathcal{Y}$
where h_i is a positive function of y_i alone.
In particular, if $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$, $Y_1 \perp\!\!\!\perp Y_2$ iff
$f(y_1, y_2) = h_1(y_1)\, h_2(y_2) \quad \forall (y_1, y_2) \in \mathcal{Y}$ where
$h_i(y_i) > 0$ for $y_i \in \mathcal{Y}_i$ and $i = 1, 2$.

Th 2.2 c) $Y_1, \dots, Y_n$ are ind iff
$f(y_1, \dots, y_n) = g_1(y_1) \cdots g_n(y_n) \quad \forall \underset{\sim}{y}$ where
$g_i(y_i) \geq 0$ is a function of y_i alone.

skip
Proof a) If the support is not a cross product,
$\exists\ \underline{y} \ni f(\underline{y}) = 0$ but $f_{Y_i}(y_i) > 0$ for $i = 1, \dots, n$
so $0 = f(\underline{y}) \neq \prod_{i=1}^{n} f_{Y_i}(y_i) > 0$.

$\mathcal{Y} \neq \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ so take $\underline{y} \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$ but $\underline{y} \notin \mathcal{Y}$

b) Proof for pdf, For pmf replace integrals by sums.

7.5

If $Y_1, \ldots, Y_n$ are ind, take $h_i(y_i) = f_{Y_i}(y_i) > 0$ for $y_i \in \mathcal{Y}_i$, $i = 1, \ldots, n$.

If $f(\underline{y}) = \prod_{i=1}^{n} h_i(y_i)$ for $\underline{y} \in \mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$, then $f(\underline{y}) = 0 = \prod_{i=1}^{n} f_{Y_i}(y_i)$ for $\underline{y} \notin \mathcal{Y}$. So if

$$f(\underline{y}) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \prod_{i=1}^{n} h_i(y_i)$$ for $\underline{y} \in \mathcal{Y}$, we are done.

Now $1 = \int \cdots \int_{\mathcal{Y}} f(\underline{y})\, d\underline{y} = \prod_{i=1}^{n} \underbrace{\int_{\mathcal{Y}_i} h_i(y_i)\, dy_i}_{a_i} = \prod_{i=1}^{n} a_i.$

For $y_i \in \mathcal{Y}_i$, get marginal pdf by integrating out $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. So

$$\underline{f_{Y_i}(y_i)} = \int_{\mathcal{Y}_n} \cdots \int_{\mathcal{Y}_{i+1}} \int_{\mathcal{Y}_{i-1}} \cdots \int_{\mathcal{Y}_1} \prod_{j=1}^{n} h(y_j)\, dy_1 \cdots dy_{i-1}\, dy_{i+1} \cdots dy_n$$

$$= h_i(y_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \int_{\mathcal{Y}_j} h_j(y_j)\, dy_j = h_i(y_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j = h_i(y_i) \frac{1}{a_i}.$$

Since $\prod_{j=1}^{n} a_j = 1$ and $a_i f_{Y_i}(y_i) = h_i(y_i)$ for $y_i \in \mathcal{Y}_i$,

$$f(\underline{y}) = \prod_{i=1}^{n} h_i(y_i) = \prod_{i=1}^{n} a_i f_{Y_i}(y_i) = \left(\prod_{i=1}^{n} a_i\right) \prod_{i=1}^{n} f_{Y_i}(y_i)$$

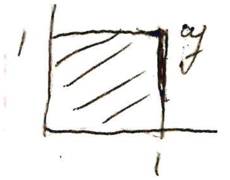
$$= \prod_{i=1}^{n} f_{Y_i}(y_i)$$ if $\underline{y} \in \mathcal{Y}$.

c) By a), the support is a cross product.
By b), take $g_i(y_i) = \begin{cases} h_i(y_i), & y_i \in \mathcal{Y}_i \\ 0, & \text{else} \end{cases}$.

skip

20] c) is taught in M483, but factorizing on the support is simpler.

ex] $f(x,y) = 4xy, \quad \underbrace{0 < x < 1, \ 0 < y < 1}_{\text{Support is a cross product}}$



$= 2x \ 2y = h_1(x)\, h_2(y)$ so $X \perp\!\!\!\perp Y$.

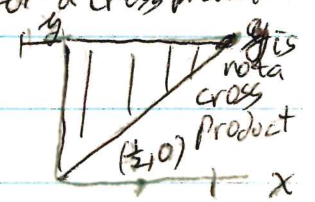
$= 4xy \ I[\{x \in (0,1)\} \cap \{y \in (0,1)\}]$

$= \underbrace{4x \ I[x \in (0,1)]}_{g_1(x)} \ \underbrace{y \ I[y \in (0,1)]}_{g_2(y)}$ using $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$.

Note $I((x,y) \in \mathcal{Y}) = I[\{x \in \mathcal{Y}_1\} \cap \{y \in \mathcal{Y}_2\}] = I(x \in \mathcal{Y}_1)\, I(y \in \mathcal{Y}_2)$ since RHS = LHS = 1 iff $x \in \mathcal{Y}_1$ and $y \in \mathcal{Y}_2$ iff $(x,y) \in \{x \in \mathcal{Y}_1\} \cap \{y \in \mathcal{Y}_2\}$.

Warning: Use indicators to denote support; the indicators must factor for a cross product.

ex] $f(x,y) = 8xy \quad \underbrace{0 < x < y < 1}_{\text{not a crossproduct}}$



so $f(x,y) = (8x)(y)$ for but X and Y are dependent.

$I(0 < x < y < 1) \neq I(x \in \mathcal{Y}_1)\, I(y \in \mathcal{Y}_2)$

ex] For HW 2 # 6 $X \perp\!\!\!\perp Y$, $Z_1 = X + Y_1$.

$$f_{X|Z_1=z}(x|z) = \frac{P(X=x, Z_1=z)}{P(Z_1=z)} = \frac{P(X=x, Y=z-x)}{P(Z_1=z)}$$

$$\overset{X \perp\!\!\!\perp Y}{=} \frac{P(X=x)\, P(Y=z-x)}{P(Z_1=z)}.$$

21] Th 2.3 $Y_1, \ldots, Y_n$ ind $\Rightarrow$ i) $Y_{i_1}, \ldots, Y_{i_k}$ are ind and

ii) $f(y_{i_1}, \ldots, y_{i_k} \mid y_{j_1}, \ldots, y_{j_m}) = f(y_{i_1}, \ldots, y_{i_k})$.

no overlap

22] * $E\, h(\underline{Y}) = \begin{cases} \int \cdots \int_{\mathbb{R}^n} h(\underline{y}) f(\underline{y})\, d\underline{y} & \text{pdf} \\ \sum \cdots \sum_{\mathbb{R}^n} h(\underline{y}) f(\underline{y}) & \text{pmf} \end{cases}$

23] Th 2.4 $E[h(Y_{i_1}, \ldots, Y_{i_k})] = \int \cdots \int_{\mathbb{R}^k} h(y_{i_1}, \ldots, y_{i_k})\, f(y_{i_1}, \ldots, y_{i_k})\, dy_{i_1} \cdots dy_{i_k}$

Replace integrals by sums for pmf

24] P38 Application: given $f(x,y)$ find marginals $f_X(x)$ and $f_Y(y)$, then 8.5

$EX = \int x f_X(x)\,dx$ and $EY = \int y f_Y(y)\,dy$.

25]* p39 If $Y_1, \ldots, Y_n$ are ind,

$E[h_1(Y_1) \cdots h_n(Y_n)] = \prod_{i=1}^{n} E h_i(Y_i) = E[h_1(Y_1)] \cdots E[h_n(Y_n)]$

proof LHS $= \int \cdots \int \prod h_i(y_i) f(\underline{y}) d\underline{y} = \int \cdots \int \prod h_i(y_i) \prod f_i(y_i) \prod dy_i = \prod \int h_i(y_i) f_i(y_i)\, dy_i = $ RHS

In particular, $Y_1 \perp\!\!\!\perp Y_2 \Rightarrow E(Y_1 Y_2) = E[Y_1]\, E[Y_2]$.

26]* p40 $\text{cov}(Y_1, Y_2) = E[(Y_1 - E(Y_1))(Y_2 - E(Y_2))]$

$= E(Y_1 Y_2) - E(Y_1) E(Y_2)$.

27] Th 2.9 If $\text{cov}(Y_1, Y_2)$ exists and $Y_1 \perp\!\!\!\perp Y_2$, $\text{Cov}(Y_1, Y_2) = 0$, but $\text{cov}(Y_1, Y_2) = 0$ does not mean that $Y_1 \perp\!\!\!\perp Y_2$.

28] common E1 problem Given table for $f(y_1, y_2)$ pmf or joint pdf $f(y_1, y_2)$ a) IS $Y_1 \perp\!\!\!\perp Y_2$?

b) Find marginals f_{Y_1}, f_{Y_2}

c) Find EY_1, EY_2, EY_1Y_2, $\text{cov}(Y_1, Y_2)$.

see ex 2.6, 2.7, 2.8.

29]* Th 2.7 If $Y_1, \ldots, Y_n$ are ind then $h_1(Y_1), \ldots, h_n(Y_n)$ are ind

cross product

ex] If $X \perp\!\!\!\perp Y$, $P(X \in A, Y \in B) = E(I[(X,Y) \mid X \in A, Y \in B])$

$= E[I_A(X) I_B(Y)] = E[I_A(X)]\, E[I_B(Y)] = P(X \in A) P(Y \in B)$

30] Know $I_A(X) \sim$ Bernoulli with $P(I_A(X) = 1) = P(X \in A)$.

§2.3 31] p41 $Y|X=x$ is a single distribution,

$Y|X$ is a family of distributions.

ex] $Y|X=x \sim N(c+dx, \sigma^2)$, $Y|X \sim N(c+dX, \sigma^2)$
is the family of normal dist's with variance σ^2
and mean $\mu_{Y|X=x} = c+dx$.

32] p42 $W \equiv W_x \equiv Y|X=x$ has pdf or pmf $f(y|x) \equiv f_W(y) \equiv f_{W_x}(y)$

The <u>conditional expected value</u>

$$E[h(Y)|X=x] = \begin{cases} \sum_y h(y) f(y|x) & \text{pmf} \\ \int_{-\infty}^{\infty} h(y) f(y|x)\,dy & \text{pdf} \end{cases}$$

In particular $E(Y|X=x) = \begin{cases} \sum_y y\, f(y|x) & \text{pmf} \\ \int_{-\infty}^{\infty} y\, f(y|x)\,dy & \text{pdf} \end{cases}$

33] p42 The conditional variance

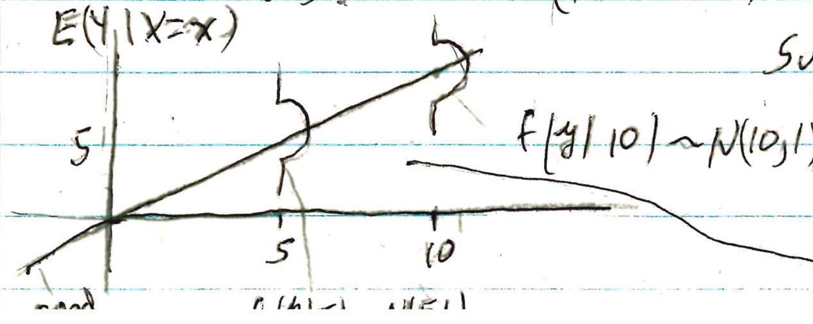
$$Var(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2$$
$$= E[(Y - \underbrace{E(Y|x)}_{\text{known constant if } X=x})^2 | X=x]$$

Note $V(Y|X=x) = E(W_x^2) - [E(W_x)]^2$

34] p42 $f(y|x)$ is a function of y with x fixed,
but $E(Y|X=x) = m(x)$ is a function of x.
$E(Y|X) = m(X)$ and $V(Y|X) = v(X)$ are
random variables.

ex] Y = Weight X = age of people in Illinois $E(Y|X=x)$ is
the expected weight of all people of age x
in Illinois. $E(Y|X=4) < E(Y|X=18)$.

ex] Suppose $Y|X=x \sim N(x,1)$
$E(Y|X=x) = x$



ex] If $p(X=5) = p(X=10) = \frac{1}{2}$ in last ex, then perform the experiment resulting in $X=x$. Sometimes $x=5$ and sometimes $x=10$.

35] Know p 43 Iterated Expectations

$$E(Y) = E[E(Y|X)]$$

$$\left(\text{or } E_Y(Y) = E_X[\underbrace{E_{Y|X}(Y|X)}_{m(X) \text{ where } m(x) = E(Y|X=x)}]\right) \text{ if the expectations exist.}$$

Proof for pdf: $E(Y) = \int\int y\, f(x,y)\, dy\, dx$

$$= \int\int y\, f_{Y|X}(y|x)\, f_X(x)\, dy\, dX = \int \left[\int y\, f_{Y|X}(y|x)\, dy\right] f_X(x)\, dx$$

$$= \int \underbrace{E(Y|X=x)}_{m(x)} f_X(x)\, dx = E[m(X)]$$

$$= E[E(Y|X)]. \quad \square$$

36] Know p43 Steiner's Formula = Conditional Variance Identity: $V(Y) = E[V(Y|X)] + V[E(Y|X)]$ provided the expected values exist.

Read proof on p 51 - 2.

37] EI problem p43 given $Y|X \sim W$ and $X \sim U$, find $E(Y)$ and $V(Y)$.

ex] See ex 2.10, 2.68 for a qual problem

ex] 2.73 $Y|P=\rho \sim \text{binomial}(k, \rho)$, $P \sim \text{beta}(\delta=4, \nu=6)$

Find a) $E(Y)$, b) $V(Y)$