soln a) $E(Y) = E[\underbrace{E(Y|P)}_{bin(k,P)}] = E(kP) = kE(P) = k\frac{\delta}{\delta+\gamma}$

$= k\frac{4}{4+6} = \boxed{0.4k}$

b) $V(Y) = E[\underbrace{V(Y|P)}_{bin(k,P)}] + V[\underbrace{E(Y|P)}_{bin(k,P)}]$

$= E[kP(1-P)] + V(kP)$

$= kE(P) - k\underbrace{E(P^2)}_{V(P)+[E(P)]^2} + k^2 V(P)$

$= k\frac{\delta}{\delta+\gamma} - k\left[\frac{\delta\gamma}{(\delta+\gamma)^2(\delta+\gamma+1)} + \left(\frac{\delta}{\delta+\gamma}\right)^2\right] + k^2\frac{\delta\gamma}{(\delta+\gamma)^2(\delta+\gamma+1)}$

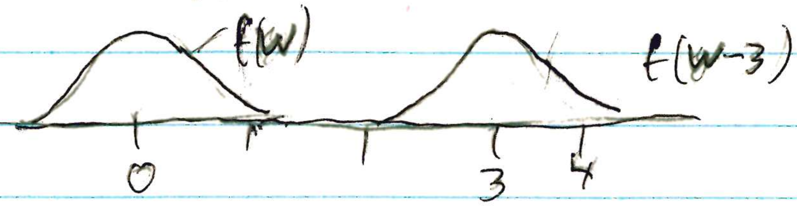
$= k0.4 - k[.021816 + .16] + k^2[.021816]$

$= .021816k^2 + .21818k$

$= \left(\frac{6}{275}k^2 + \frac{12}{55}k\right)$

↑Know for Quiz 2

§2.4 38] p44 The family of pdfs $f(w-\mu)$ indexed by $\mu \in \mathbb{R}$ is a location family for the RV $W = Y + \mu$ with standard pdf $f(w)$ $(\mu = 0)$.



μ is the location parameter

39 p44 The family of pdfs $\frac{1}{\sigma}f\left(\frac{w}{\sigma}\right)$ indexed by scale parameter $\sigma > 0$ is a scale family for the RV $W = \sigma Y$ with standard pdf $f(w)$.

40) p44 The family of pdfs $\frac{1}{\sigma} f\left(\frac{w-\mu}{\sigma}\right)$ indexed by location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$ is the <u>location scale family</u> for RV $W = \mu + \sigma Y$ with standard pdf $f(w)$.

ex] $N(\mu, \sigma^2) \to \mu, \sigma$, $\qquad Y \sim N(0,1)$
Cauchy$(\mu, \sigma) \to \mu, \sigma$, $\qquad Y \sim C(0,1)$
uniform $(\theta_1, \theta_2) \to \mu = \theta_1$, $\sigma = \theta_2 - \theta_1$, $\qquad Y \sim U(0,1)$
and see p 44

41] Let Y have pdf $f_Y(y)$ and cdf $F_Y(y)$.

i) If $W = \mu + Y$, then
$$F_W(y) = P(W \le y) = P(Y \le y - \mu) = F_Y(y-\mu)$$
and $f_W(y) = \frac{d}{dy} F_W(y) = f_Y(y-\mu)$.

ii) If $W = \sigma Y$, $\sigma > 0$, then
$$F_W(y) = P(W \le y) = P\left(Y \le \frac{y}{\sigma}\right) = F_Y\left(\frac{y}{\sigma}\right)$$
and $f_W(y) = \frac{d}{dy} F_W(y) = \frac{1}{\sigma} f_Y\left(\frac{y}{\sigma}\right)$.

iii) If $W = \mu + \sigma Y$, $\sigma > 0$, then $P(W \le y)$
$$= P(\mu + \sigma Y \le y) = P\left(Y \le \frac{y-\mu}{\sigma}\right) = F_Y\left(\frac{y-\mu}{\sigma}\right)$$

and $f_W(y) = \frac{d}{dy} F_W(y) = \frac{1}{\sigma} f_Y\left(\frac{y-\mu}{\sigma}\right)$.

Application: A table of probabilities for standard RV Y also works for $W = \mu + \sigma Y$.
$Y = Z \sim N(0,1)$ for normal Z table.

§2.5 41] p45 If X is a discrete RV with support $\mathcal{X}$ and $Y = t(X)$, then the pmf of Y is $f_Y(y) = \sum_{x \in \mathcal{X} \mid t(x) = y} f_X(x)$

42] Know for E1 Given a table for $f_X(x)$ compute $y = t(x)$ and collect terms

$y = x^2$	1	0	1
x	-1	0	1
$f_X(x)$	.1	.4	.5

$Y = X^2$ has

y	0	1
$f_Y(y)$	.4	.6

43] p45 h is increasing if $y_1 < y_2 \Rightarrow h(y_2) > h(y_1)$

Oct 2.19 decreasing $h(y_2) < h(y_1)$

monotone if h is increasing or decreasing.

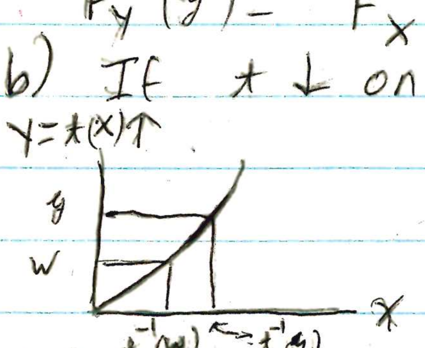
44] p46 ✱ Method of Distributions Let X and $Y = t(X)$ have pdfs.

Let $\mathcal{X} = \{x \mid f_X(x) > 0\}$ and $\mathcal{Y} = \{y \mid y = t(x) \text{ for some } x \in \mathcal{X}\}$.

a) If $t \uparrow$ on $\mathcal{X}$, then

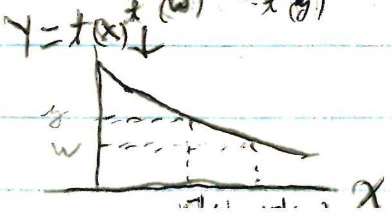
$F_Y(y) = F_X(t^{-1}(y))$ for $y \in \mathcal{Y}$.

b) If $t \downarrow$ on $\mathcal{X}$, then $F_Y(y) = 1 - F_X(t^{-1}(y))$ for $y \in \mathcal{Y}$.



$\{w \in \mathcal{Y} \mid w \le y\} = \{x \in \mathcal{X} \mid t(x) \le y\}$

$= \{x \in \mathcal{X} \mid x \le t^{-1}(y)\}$



$\{w \in \mathcal{Y} \mid w \le y\} = \{x \in \mathcal{X} \mid t(x) \le y\}$

$= \{x \in \mathcal{X} \mid x \ge t^{-1}(y)\}$

proof a) $P(Y \le y) = P(t(X) \le y) = P(X \le t^{-1}(y)) \overset{11.5}{=} F_X(t^{-1}(y))$

b) $P(Y \le y) = P(t(X) \le y) = \displaystyle\int_{\{x \mid t(x) \le y\}} f_X(x)\,dx$

$= \displaystyle\int_{\{x:\ x \ge t^{-1}(y)\}} f_X(x)\,dx = \int_{t^{-1}(y)}^{\infty} f_X(x)\,dx = P[X \ge t^{-1}(y)]$

inequality reverses if $t\downarrow$ (eg $Y = -X \downarrow$ $-X \le y \Leftrightarrow x \ge -y = t^{-1}(y)$)

$= 1 - P[X \le t^{-1}(y)] = 1 - F_X[t^{-1}(y)]$.

45] If $Y = X^2$ has a pdf and $\mathcal{X} \subseteq (0, \infty)$, then $X^2 \uparrow$
$\subseteq (-\infty, 0)$ $X^2 \downarrow$.

If $Y = X^2$ and $\mathcal{X} = (-a, a)$, $a = \infty$ possible,
then $F_Y(y) = P(Y \le y) = P(X^2 \le y) =$
$P(-\sqrt{y} \le X \le \sqrt{y}) = \displaystyle\int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx =$ ← $\int_a^b f_X(x)dx = F(b) - F(a)$

$F_X(\sqrt{y}) - F_X(-\sqrt{y})$, $0 \le y \le a^2$.

46] Know for E1 Th 2.13 Transformation method for pdfs.
Let X have pdf $f_X(x)$ and support $\mathcal{X}$.
Let $Y = t(X)$ where $t\uparrow$ or $t\downarrow$ on $\mathcal{X}$.
Suppose $t^{-1}(y)$ has continuous derivative on $\mathcal{Y}$.

sometimes on qual

(*) Then $f_Y(y) = f_X(t^{-1}(y)) \left| \dfrac{d\,t^{-1}(y)}{dy} \right|$ for $y \in \mathcal{Y}$.

proof use chain rule on $F_Y(y)$ from 44].

$\dfrac{d}{dy} F_X(t^{-1}(y)) = f_X(t^{-1}(y)) \dfrac{d\,t^{-1}(y)}{dy}$ if $t\uparrow$

$\dfrac{d}{dy} \left(1 - F_X(t^{-1}(y))\right) = f_X(t^{-1}(y)) \left(- \dfrac{d\,t^{-1}(y)}{dy}\right)$ if $t\downarrow$

but $\left|\dfrac{d\,t^{-1}(y)}{dy}\right| = \begin{cases} \dfrac{d\,t^{-1}(y)}{dy} & t\uparrow \\ -\dfrac{d\,t^{-1}(y)}{dy} & t\downarrow \end{cases}$

47] * Tips p 48 i) simplify (*) as much as possible.

ii) To find $x = t^{-1}(y)$, solve $y = t(x)$ for x.

iii) Usually $\mathcal{X}$ is an interval with endpoints $a < b$ ($a = -\infty$ $b = \infty$ possible)

Let $t(a) = \lim_{x \downarrow a} t(x)$ and $t(b) = \lim_{x \uparrow b} t(x)$.

Then $\mathcal{Y}$ is an interval with endpoints $t(a)$ and $t(b)$.

iv) Often Y is gamma or exponential.

v) $y = \underbrace{\log(x)}_{\ln(x)}$ or $y = e^x$ are common.

ex] See ex. 2.14. Problems 2.16, 2.18a 2.25b have solutions

ex] ch2HW & ch10 see $W = \log(Y) \sim EXP(\lambda)$ etc

ex] $f_X(x) = 3x^2$, $0 < x < 1$. Find the pdf of $Y = 1 - X^2$.

Soln step i) $y = t(x) = 1 - x^2$, $t(0) = 1$, $t(1) = 0$ so $\mathcal{Y} = (0,1)$

ii) $y = 1 - x^2$ or $x^2 = 1 - y$ or $x = \sqrt{1-y} = t^{-1}(y) = (1-y)^{1/2}$

iii) $\left| \frac{d}{dy} t^{-1}(y) \right| = \left| \frac{1}{2}(1-y)^{-\frac{1}{2}}(-1) \right| = \frac{1}{2\sqrt{1-y}}$ on $\mathcal{Y}$.

iv) $f_Y(y) = f_X(t^{-1}(y)) \left| \frac{d\, t^{-1}(y)}{dy} \right|$

$= 3(\sqrt{1-y})^2 \frac{1}{2\sqrt{1-y}} = \underbrace{\frac{3}{2}\sqrt{1-y}, \quad 0 < y < 1}$

$f_Y(y)$ and support need to be correct.

48] p49–52 Bivariate Transformation Method

Let X_1 and X_2 have joint pdf $g(x_1, x_2)$ with support $\mathcal{X}$. Let $Y_i = t_i(X_1, X_2)$ for $i = 1, 2$

and assume that the transformation can be solved for $x_1 = t_1^{-1}(y_1, y_2)$ and $x_2 = t_2^{-1}(y_1, y_2)$.

The Jacobian is

$$J = \det \begin{bmatrix} \frac{\partial t_1^{-1}}{\partial y_1} & \frac{\partial t_1^{-1}}{\partial y_2} \\ \frac{\partial t_2^{-1}}{\partial y_1} & \frac{\partial t_2^{-1}}{\partial y_2} \end{bmatrix} \quad \text{where}$$

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Let

$|J|$ be the absolute value of J.

Then $f(y_1, y_2) = g\left(t_1^{-1}(y_1, y_2),\ t_2^{-1}(y_1, y_2)\right)|J|$

for $y \in \mathcal{Y}$.

Tip for finding $\mathcal{Y}$: $\prod_{j=1}^{K} I_{A_j}(\underline{y}) = I_{\bigcap_{j=1}^{K} A_j}(\underline{y})$ since LHS $=1$ iff all $I_{A_j}(\underline{y}) = 1$ iff RHS $=$

ex] See ex 2.17

ex] $X_1 \perp\!\!\!\perp X_2$ and interest is in $Y_1 = X_1 + X_2$.

Take $Y_2 = X_1$ so J is simple.

Then $X_1 = Y_2 = t_1^{-1}(y_1, y_2)$ and

$X_2 = Y_1 - Y_2 = t_2^{-1}(y_1, y_2)$.

$$\frac{\partial t_1^{-1}}{\partial y_1} = 0 \qquad \frac{\partial t_1^{-1}}{\partial y_2} = 1 \qquad J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$$\frac{\partial t_2^{-1}}{\partial y_1} = 1 \qquad \frac{\partial t_2^{-1}}{\partial y_2} = -1$$

$|J| = 1$ so $f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}\left(t_1^{-1}(\underline{y}), t_2^{-1}(\underline{y})\right) \underbrace{|J|}_{1}$

$= f_{X_1, X_2}(y_2, y_1 - y_2) = f_{X_1}(y_2)\, f_{X_2}(y_1 - y_2)$.

convolution formula

$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_{-\infty}^{\infty} f_{X_1}(y_2)\, f_{X_2}(y_1 - y_2)\, dy_2$

49] <u>Know for E1</u> Be able to do a bivariate transformation and to find $\mathcal{Y}$.

ex] ex2.16 $f_{X_1,X_2}(x_1,x_2) = 2e^{-(x_1+x_2)}, \quad 0 < x_1 < x_2 < \infty.$

$Y_1 = X_1, \quad Y_2 = X_1 + X_2$. So $x_1 = y_1 = t_1^{-1}(y_1, y_2)$

and $x_2 = y_2 - y_1 = t_2^{-1}(y_1, y_2)$

$\frac{\partial t_1^{-1}}{\partial y_1} = 1 \qquad \frac{\partial t_1^{-1}}{\partial y_2} = 0$

$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 = |J|$

$\frac{\partial t_2^{-1}}{\partial y_1} = -1 \qquad \frac{\partial t_2^{-1}}{\partial y_2} = 1$

Important step is going from

$\mathcal{X} = \{(x_1,x_2) \mid 0 < x_1 < x_2 < \infty\}$ to $\mathcal{Y}$.

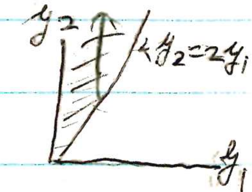
$x_1 < x_2 \Rightarrow y_1 < y_2 - y_1$ or $2y_1 < y_2$.

So $\mathcal{Y} = \{(y_1, y_2) \mid 0 < 2y_1 < y_2\}$

Since $f_{X_1,X_2}(x_1,x_2) = 2e^{-(x_1+x_2)} I(0 < x_1 < x_2 < \infty)$,

$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(t_1^{-1}(\underline{y}), t_2^{-1}(\underline{y})) = 2e^{-(y_1+y_2-y_1)} I(0 < y_1 < y_2 - y_1)$

$= 2e^{-y_2} I(0 < 2y_1 < y_2)$.



Q] Are Y_1 & Y_2 ind? $\boxed{A}$ NO !

marginals $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} 2e^{-y_2} I(0 < 2y_1 < y_2)\, dy_2 = \int_{2y_1}^{\infty} 2e^{-y_2}\, dy_2$

take deriv wrt y_2 → $= -2e^{-y_2}\Big|_{2y_1}^{\infty} = 0 - (-2e^{-2y_1}) = 2e^{-2y_1}, \quad 0 < y_1 < \infty$

$f_{Y_2}(y_2) = \int_0^{y_2/2} 2e^{-y_2}\, dy_1 = 2e^{-y_2} y_1 \Big|_{y_1=0}^{y_1=y_2/2} = y_2 e^{-y_2}, \quad 0 < y_2 < \infty$

$\forall\ y_1 = \frac{y_2}{2}$

(take deriv wrt y_1)

50] Often $Y_1 = t_1(X_1, X_2)$ is of interest. 13.5
Define $Y_2 = t_2(X_1, X_2)$ so finding
$t_1^{-1}(\underline{y}), t_2^{-1}(\underline{y})$ and J is easy. Get

the joint pdf $f_{Y_1,Y_2}(y_1, y_2)$ and find the

marginal $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2)\, dy_2$.

end exam1 material

§2.6 51] Th 2.15 p 52 $E(a) = a$

$E[aY] = a\,E(Y)$

$V(aY) = a^2 V(Y)$

$E\left[\sum_{i=1}^{k} g_i(Y_1, \ldots, Y_n)\right] = \sum_{i=1}^{k} E[g_i(Y_1, \ldots, Y_n)]$.

Let $W_1 = \sum_{i=1}^{n} a_i Y_i$ and $W_2 = \sum_{i=1}^{m} b_i X_i$.

$E(W_1) = \sum_{i=1}^{n} a_i E(Y_i)$.

(*) $V(W_1) = cov(W_1, W_1) = \sum_{i=1}^{n} a_i^2 V(Y_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j cov(Y_i, Y_j)$

$Cov(W_1, W_2) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j cov(Y_i, X_j)$

52] Know p 53 i) $E \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} EY_i$

ii) If $Y_1, \ldots, Y_n$ are ind, $V\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} V(Y_i)$.

53] * p 53 A <u>statistic</u> is a function of the data $Y_1, \ldots, Y_n$ that does not depend on any unknown parameters